CS103
Fall 2025

Lecture 10:

Graph Theory

Part 2 of 3

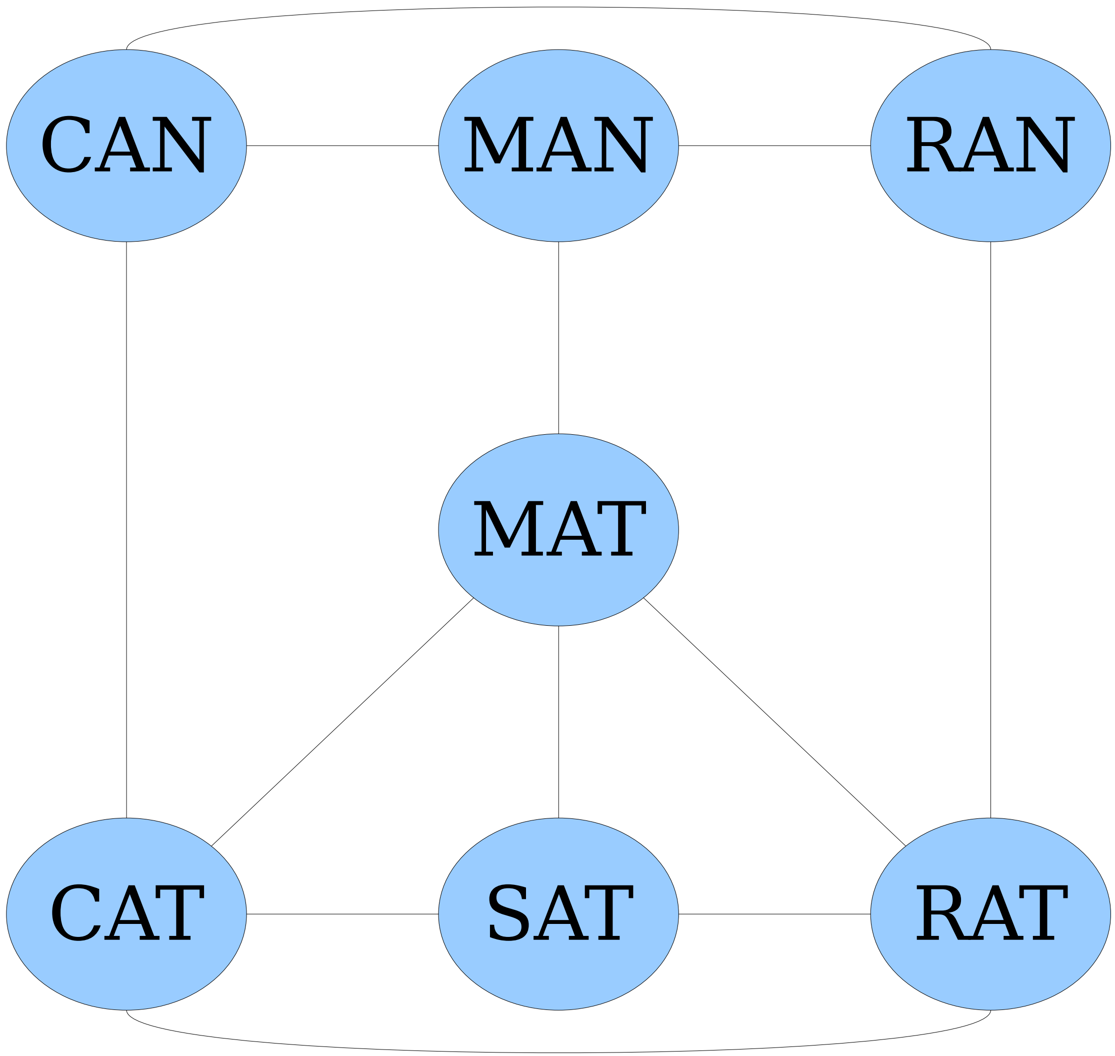
Outline for Today

- ***Walks, Paths, and Reachability***
  - Walking around a graph.
- ***Application: Local Area Networks***
  - Graphs meet computer networking.
- ***Trees***
  - A fundamental class of graphs.

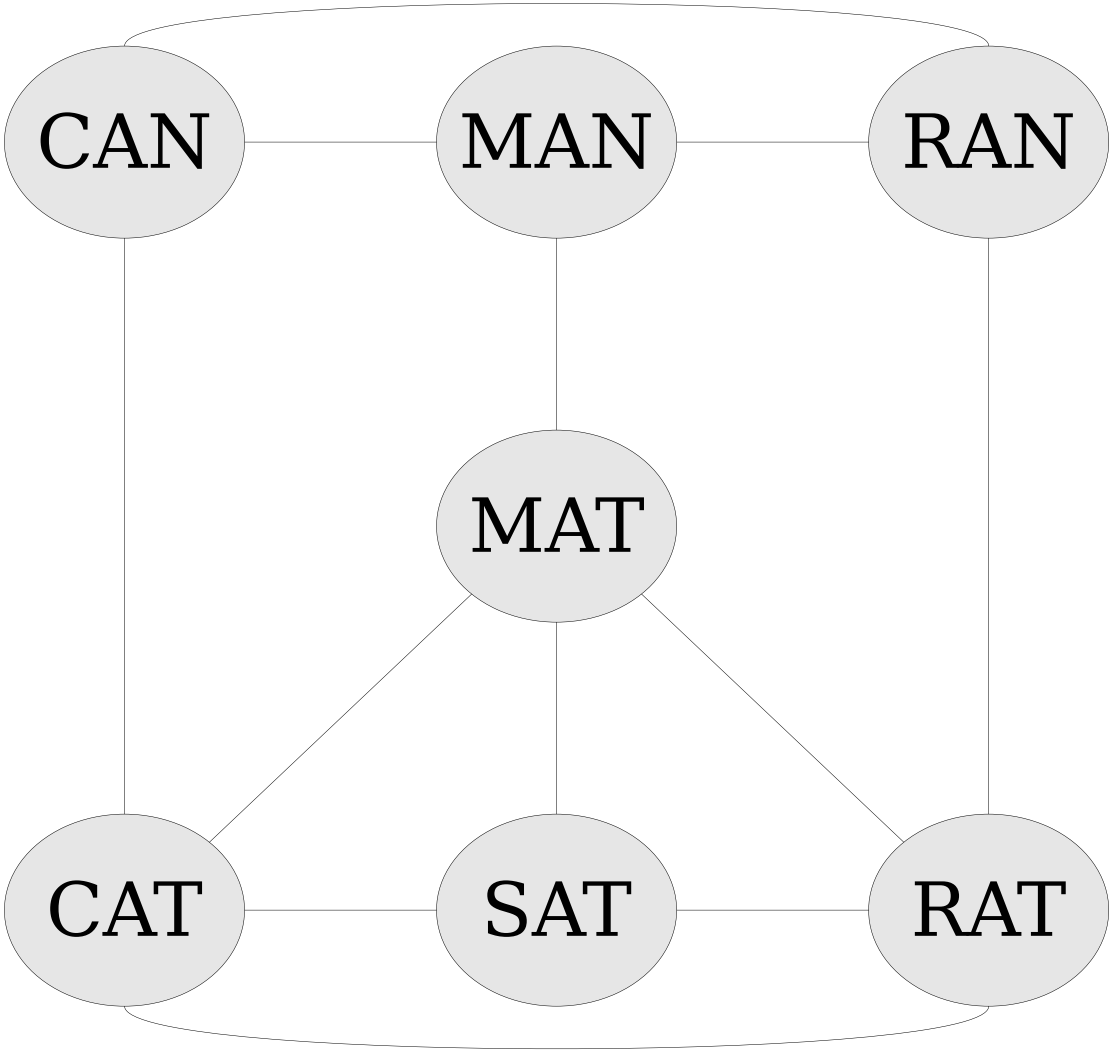
Recap from Last Time

Graphs and Digraphs

- A ***graph*** is a pair $G = (V, E)$ of a set of nodes V and set of edges E.
  - Nodes can be anything.
  - Edges are ***unordered pairs*** of nodes. If $\{u, v\} \in E$, then there's an edge from u to v.
- A ***digraph*** is a pair $G = (V, E)$ of a set of nodes V and set of directed edges E.
  - Each edge is represented as the ordered pair (u, v) indicating an edge from u to v.



Two nodes in an undirected graph are called ***adjacent*** if there is an edge between them.



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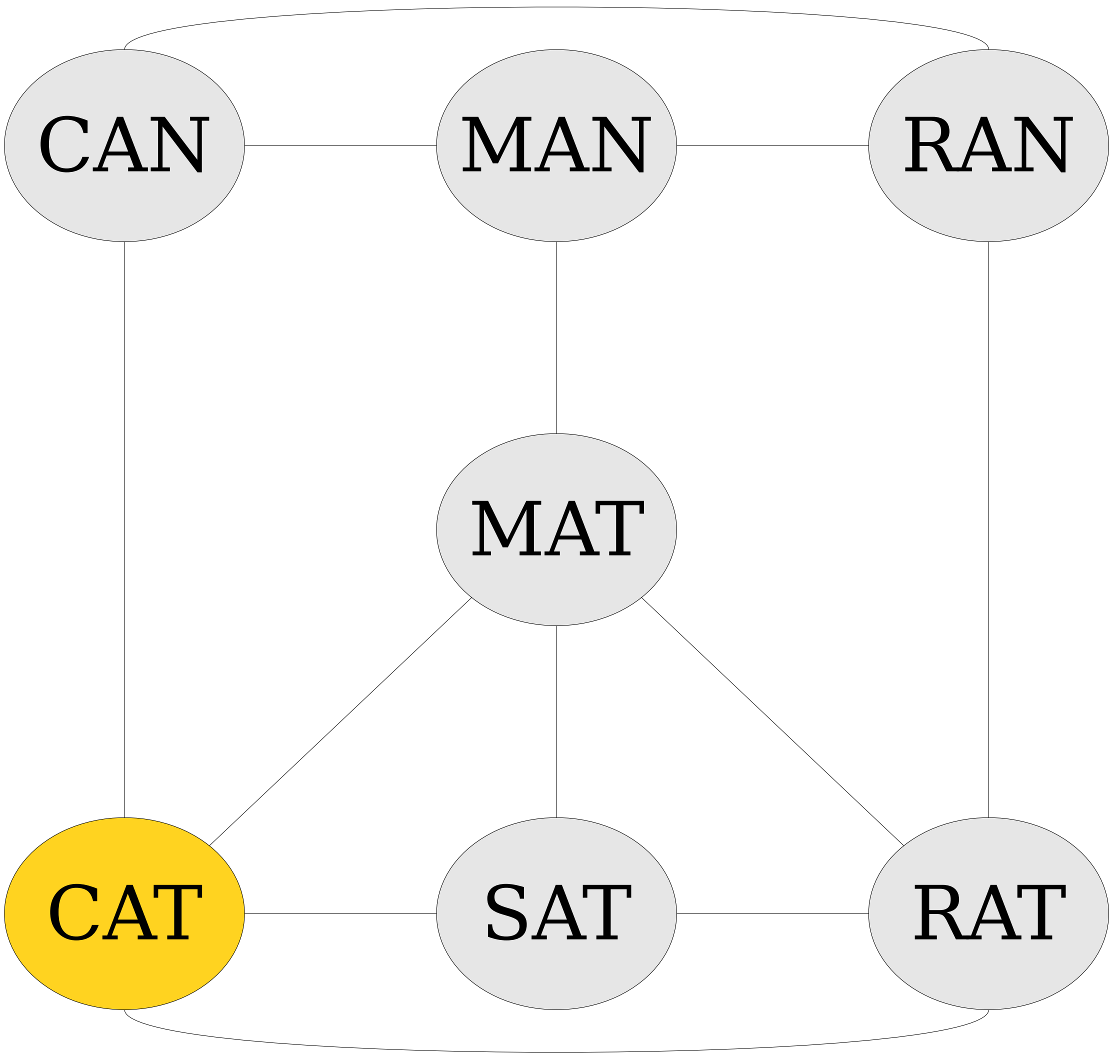

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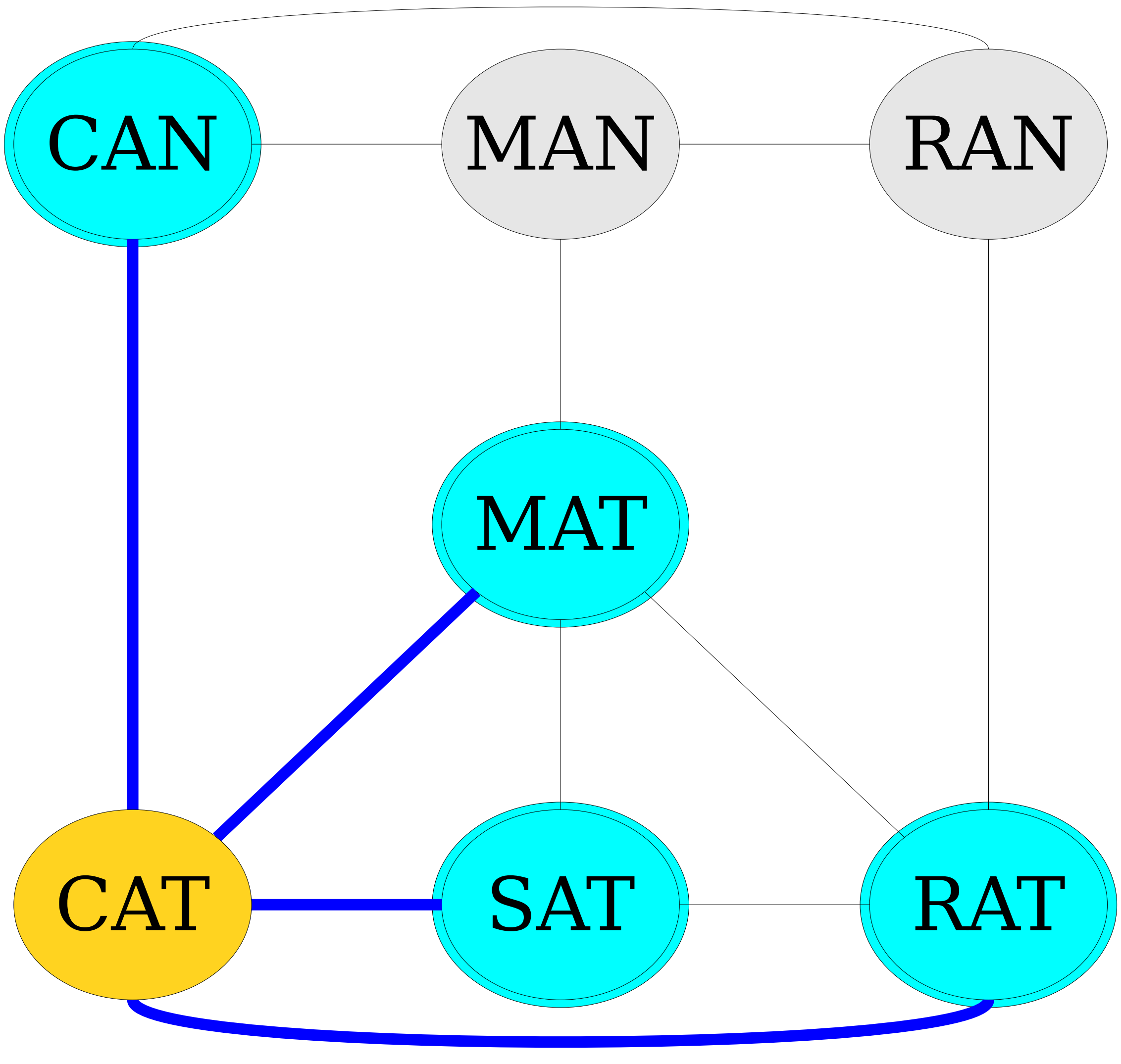


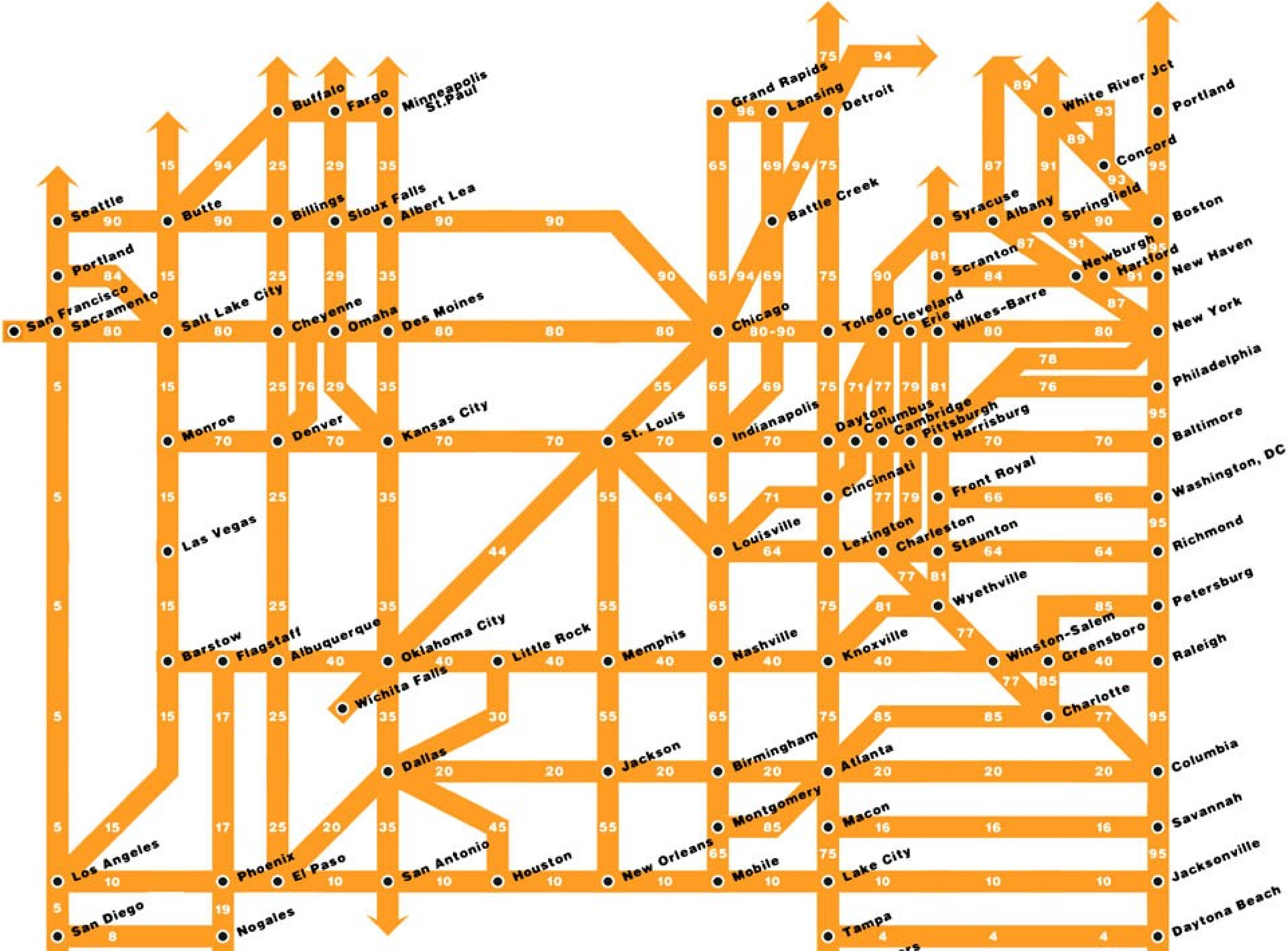
Two nodes in an undirected graph are called ***adjacent*** if there is an edge between them.

Using our Formalisms

- Let $G = (V, E)$ be an (undirected) graph.
- Intuitively, two nodes are adjacent if they're linked by an edge.
- Formally speaking, we say that two nodes $u, v \in V$ are ***adjacent*** if we have $\{u, v\} \in E$.
- There isn't an analogous notion for directed graphs. We usually just say "there's an edge from u to v" as a way of reading $(u, v) \in E$ aloud.

New Stuff!

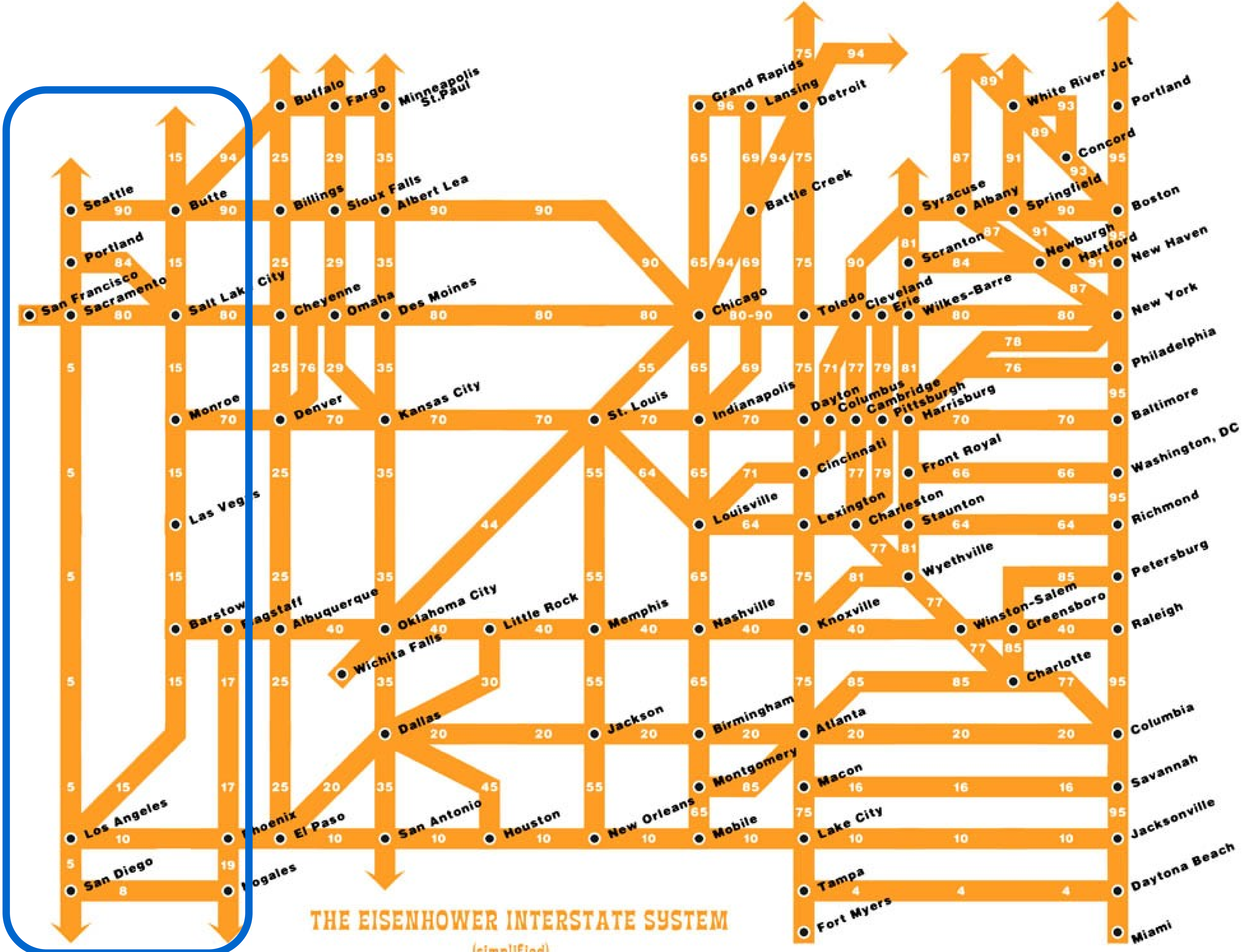
Walks, Paths, and Reachability



THE EISENHOWER INTERSTATE SYSTEM

(simplified)

CHRIS YATES 2007



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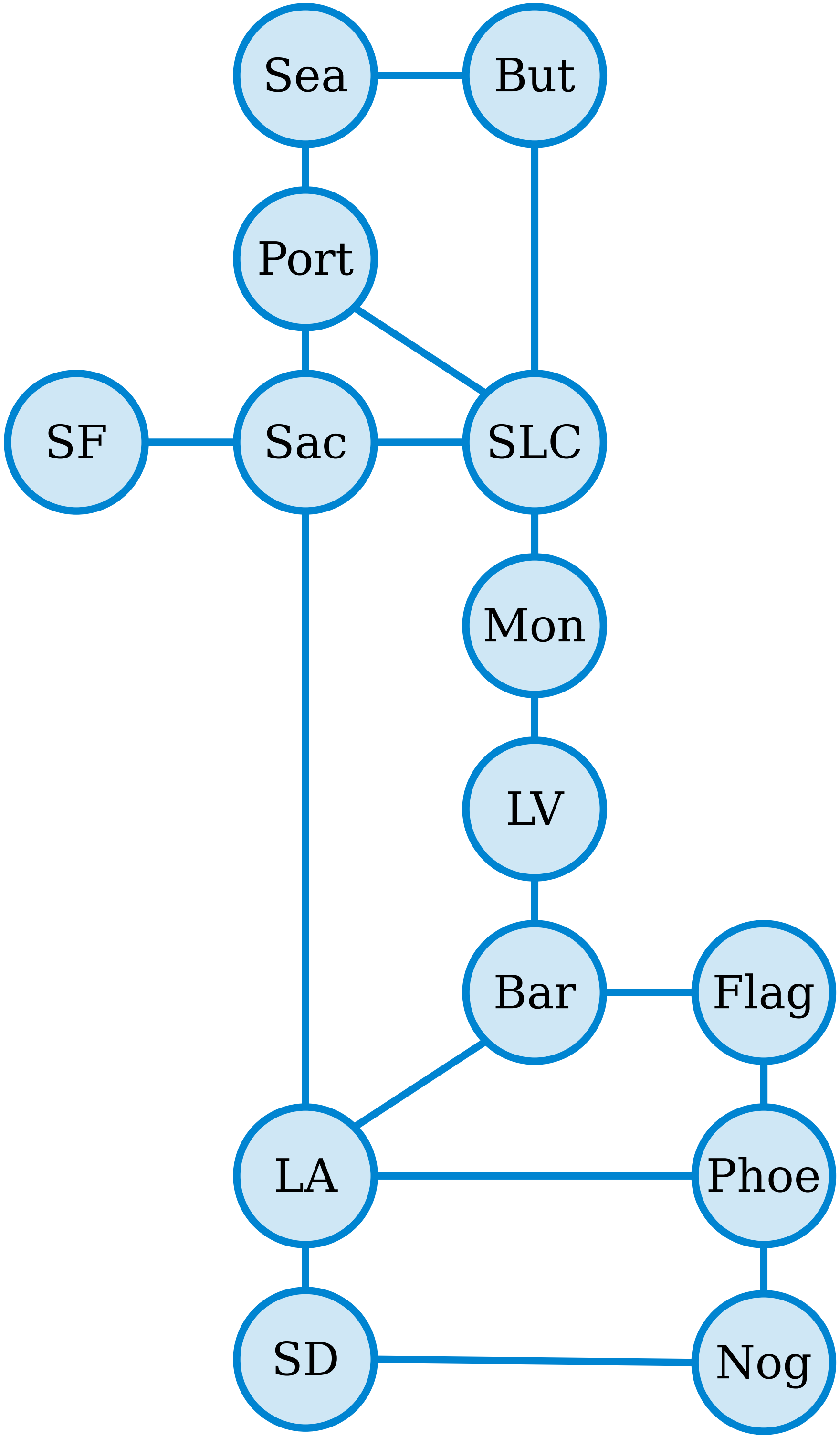
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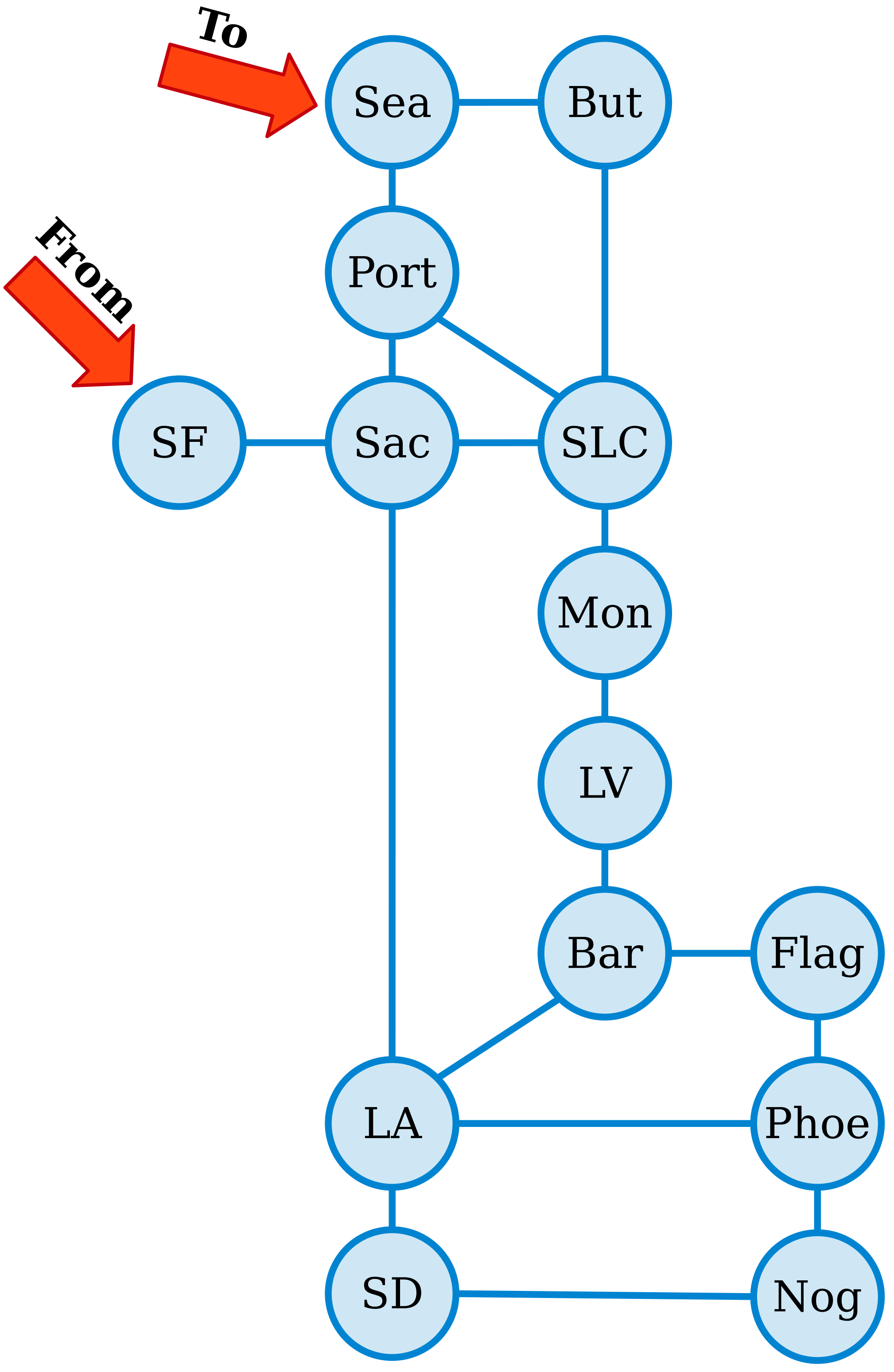
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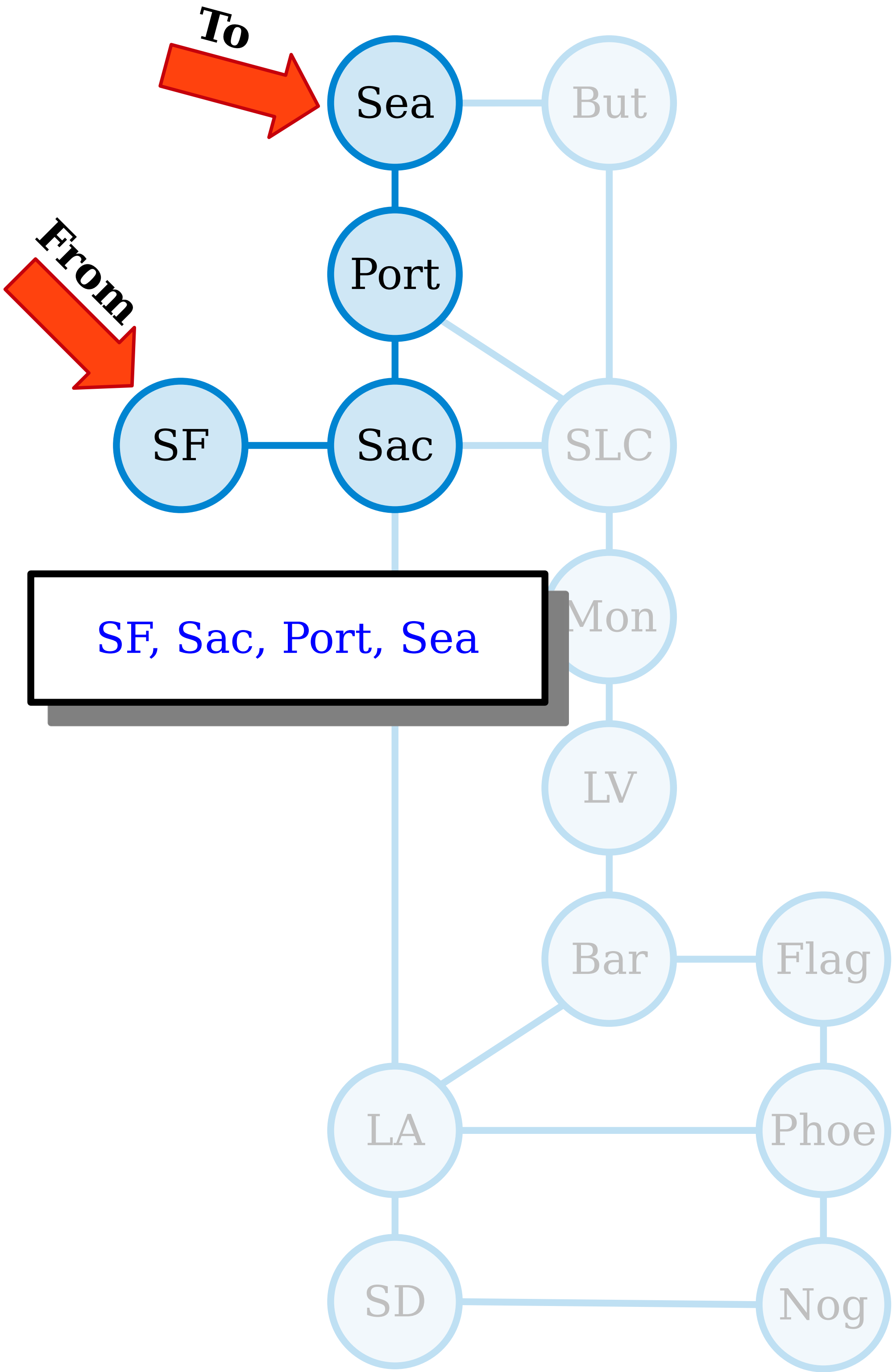
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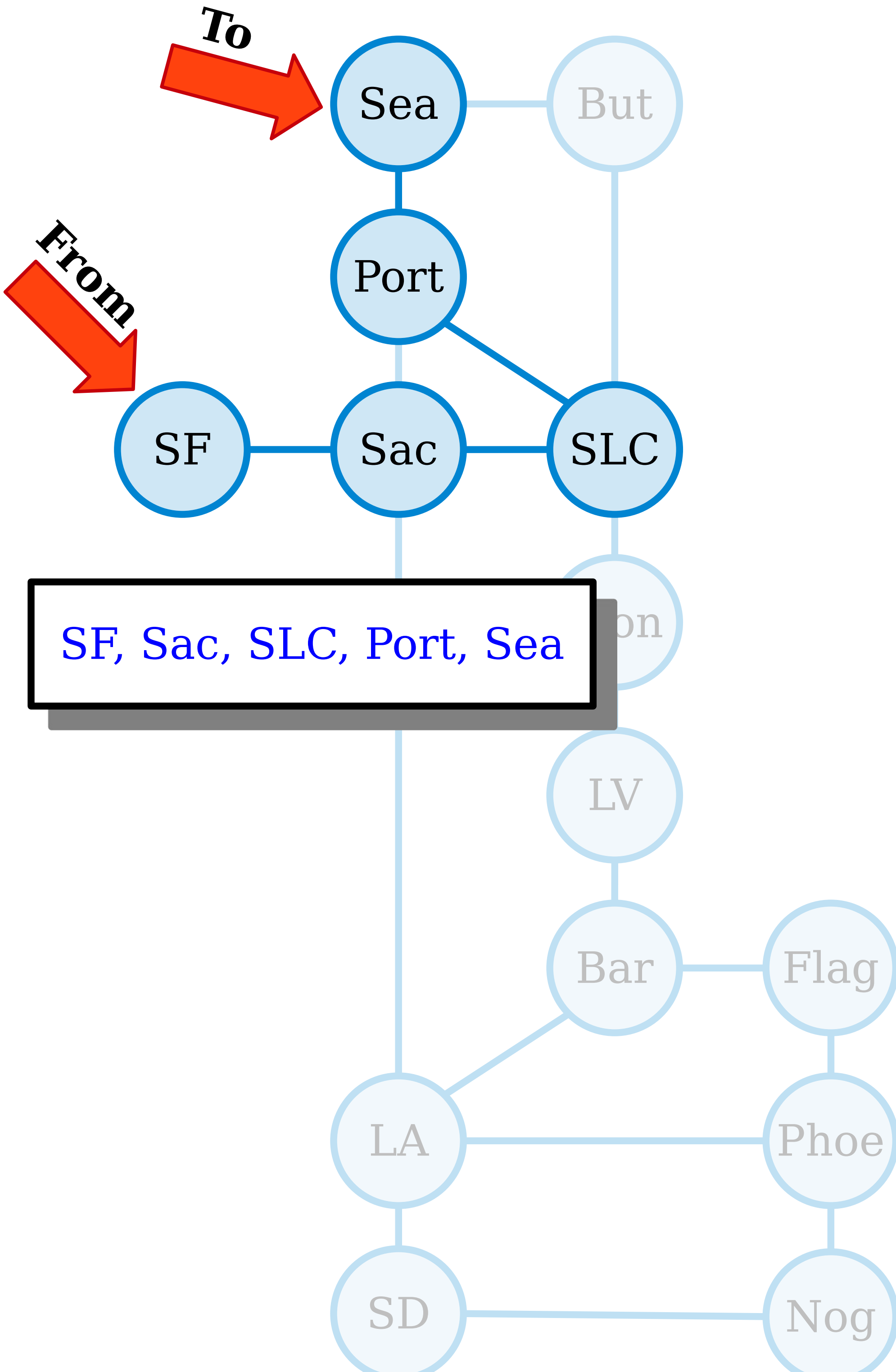
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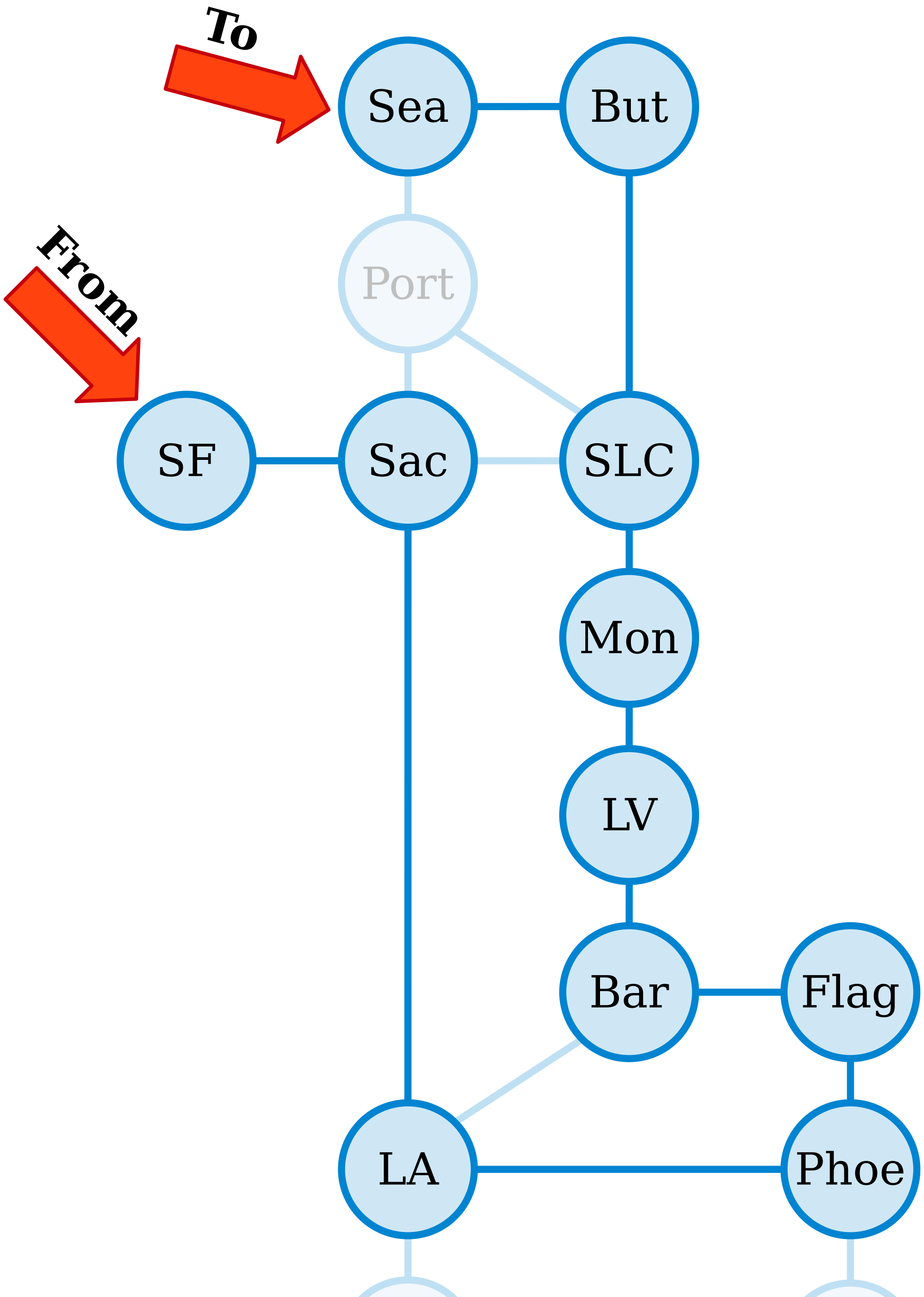
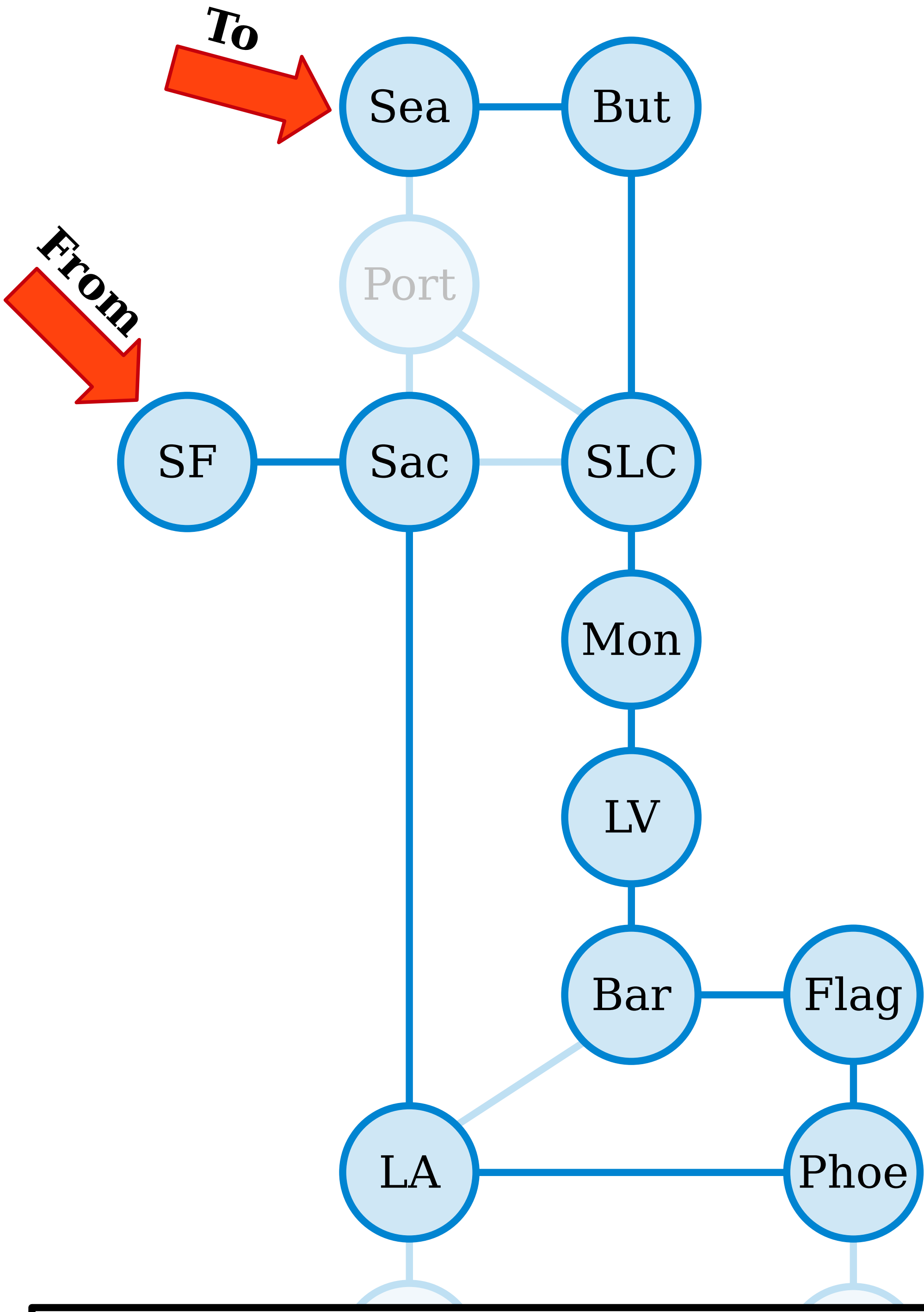
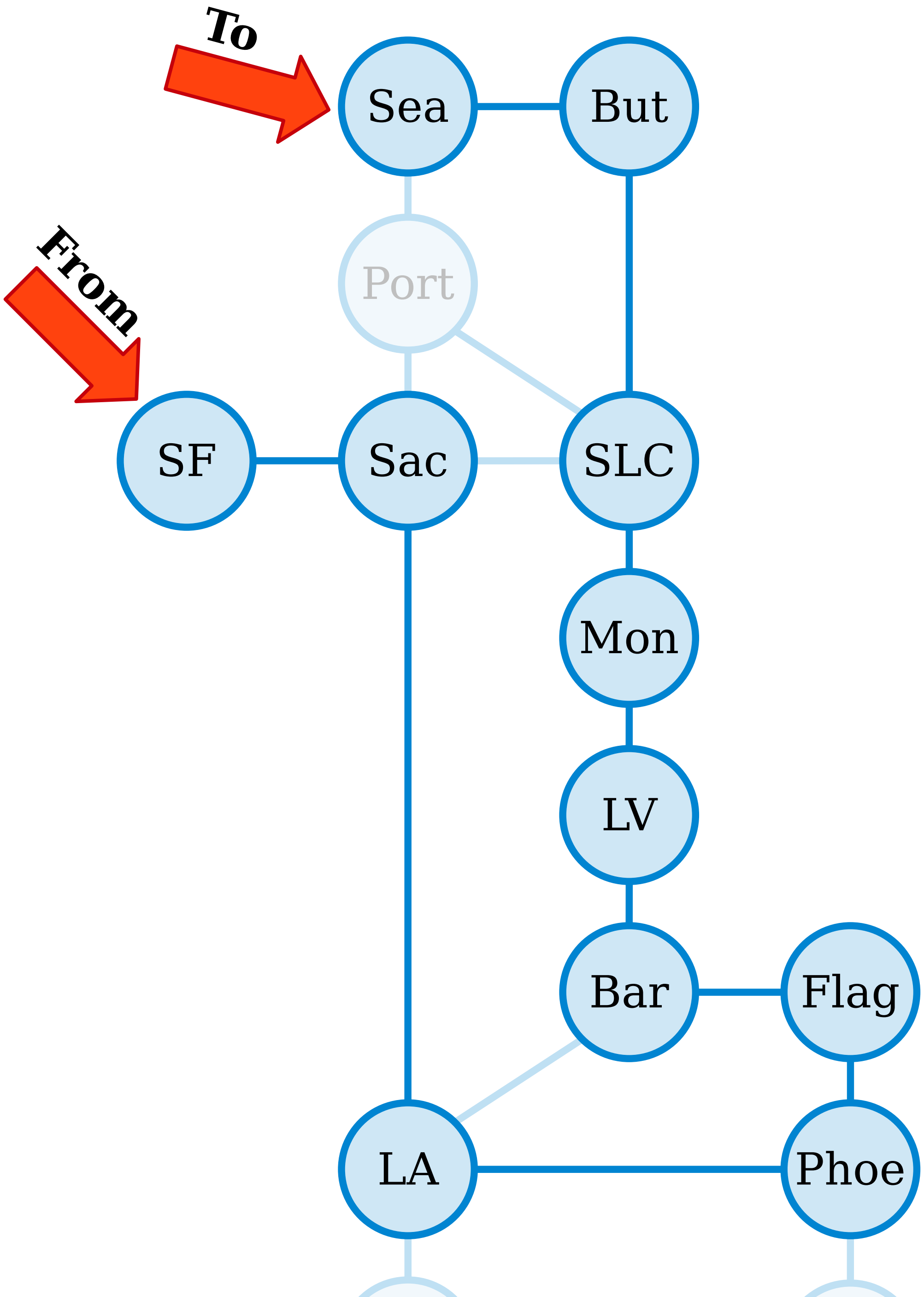


SF, Sac, LA, Phoe, Flag, Bar, LV, Mon, SLC, But, Sea



A ***walk*** in a graph $G = (V, E)$ is a sequence of one or more nodes $v_1, v_2, v_3, \ldots, v_n$ such that any two consecutive nodes in the sequence are adjacent.

SF, Sac, LA, Phoe, Flag, Bar, LV, Mon, SLC, But, Sea



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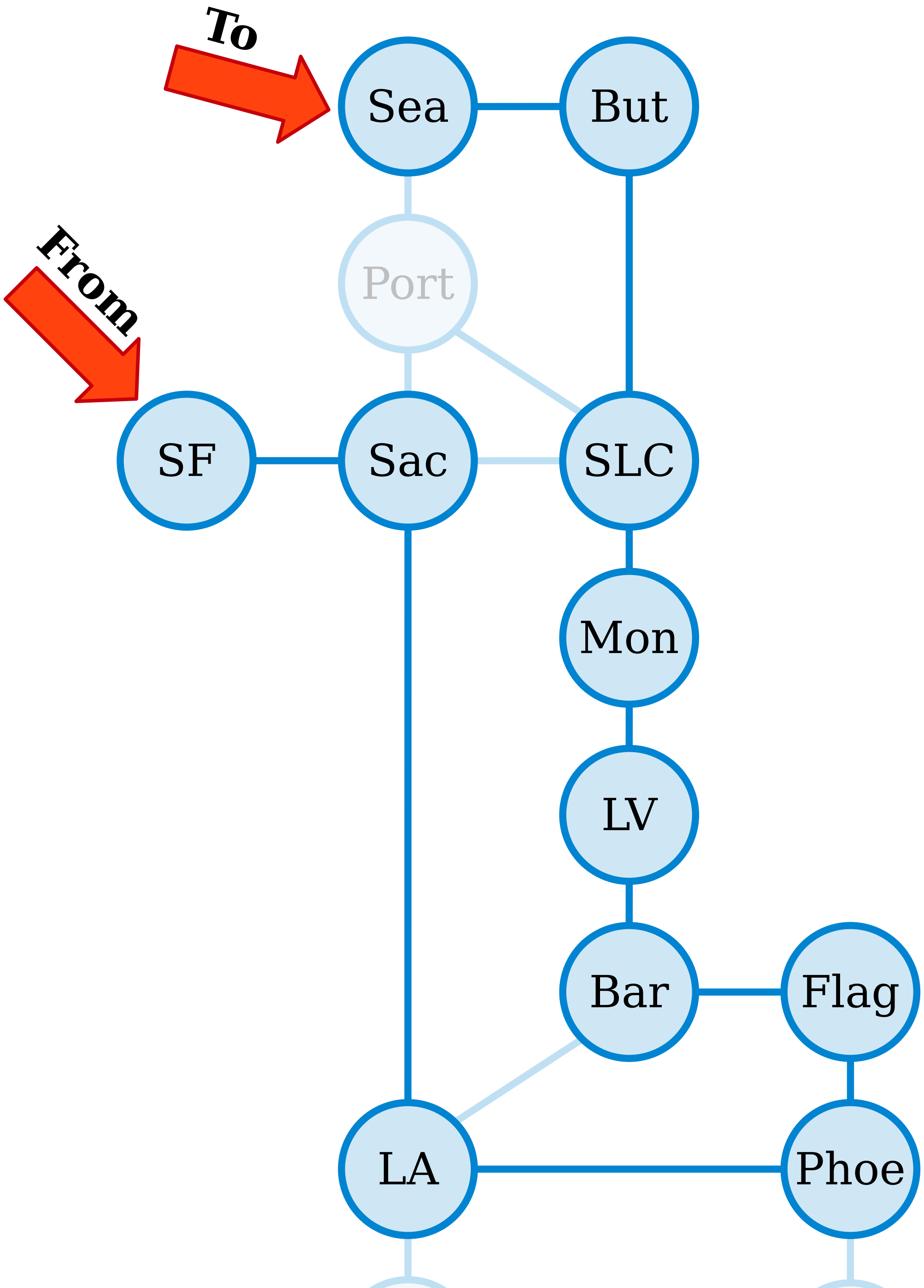

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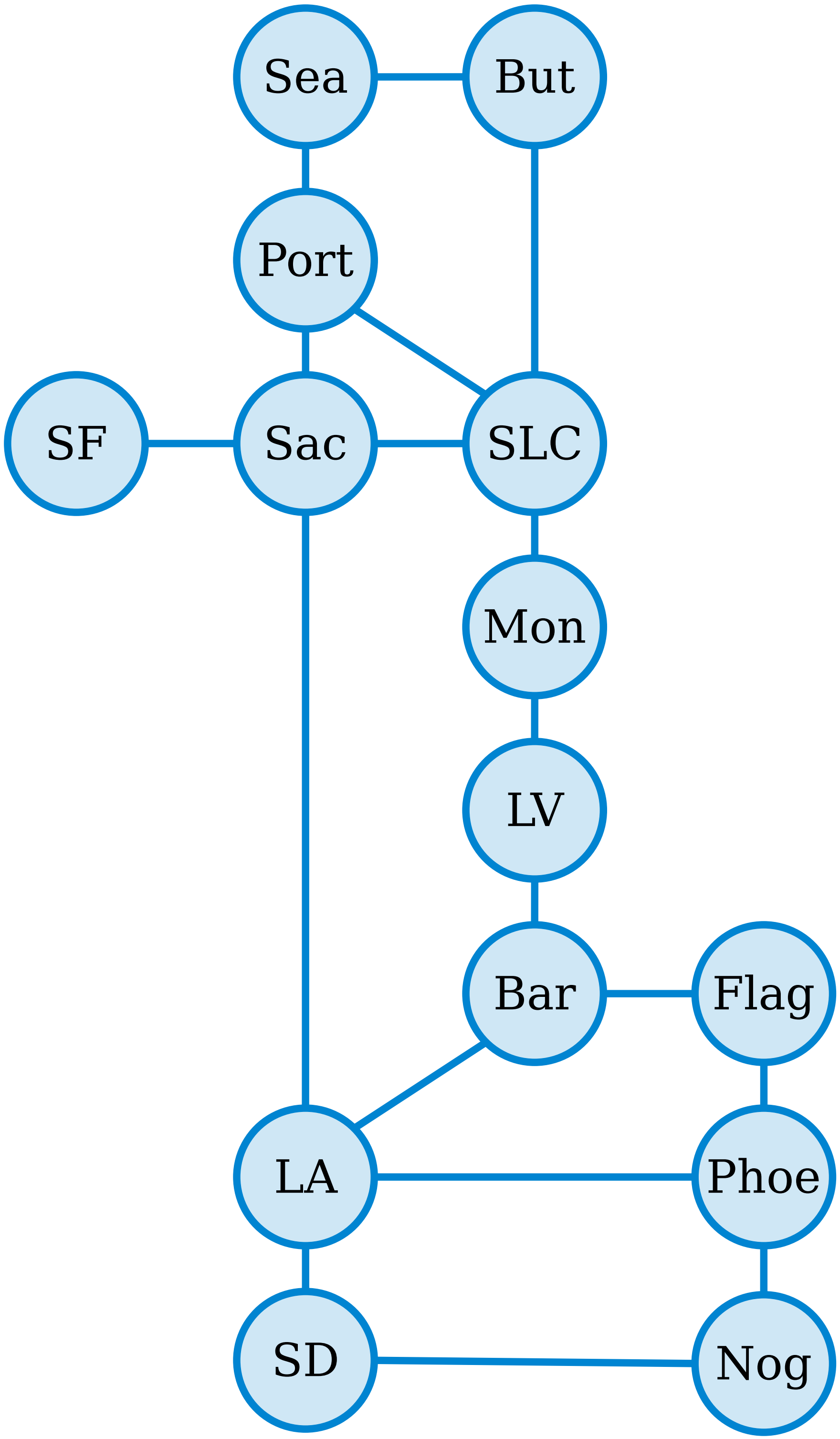
(This walk has length 10, but visits 11 cities.)

SF, Sac, LA, Phoe, Flag, Bar, LV, Mon, SLC, But, Sea



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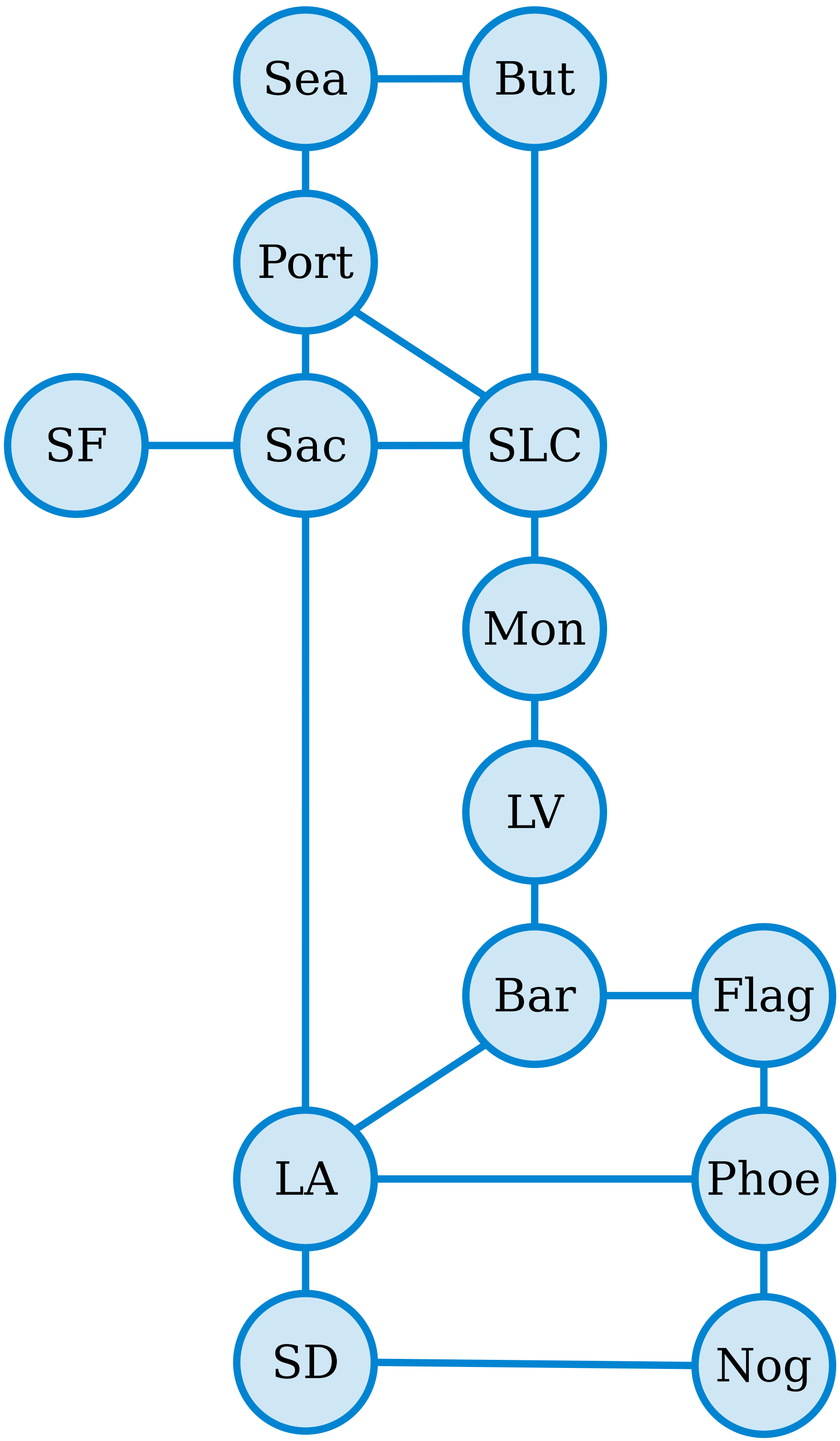

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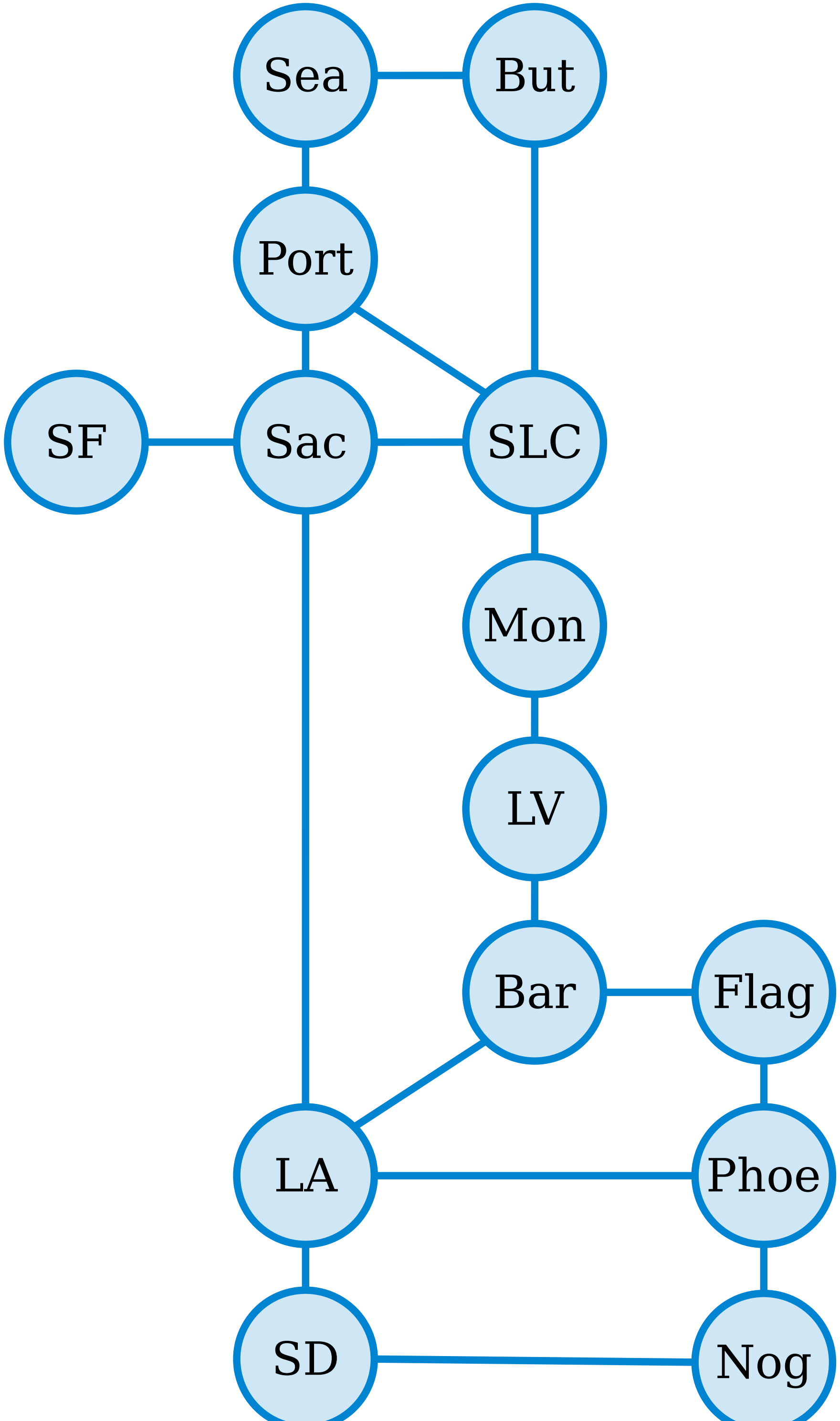
Which of these are walks in this graph?

SF
SF, Sac
SF, Sac, SF

Answer at
https://cs103.stanford.edu/pollev



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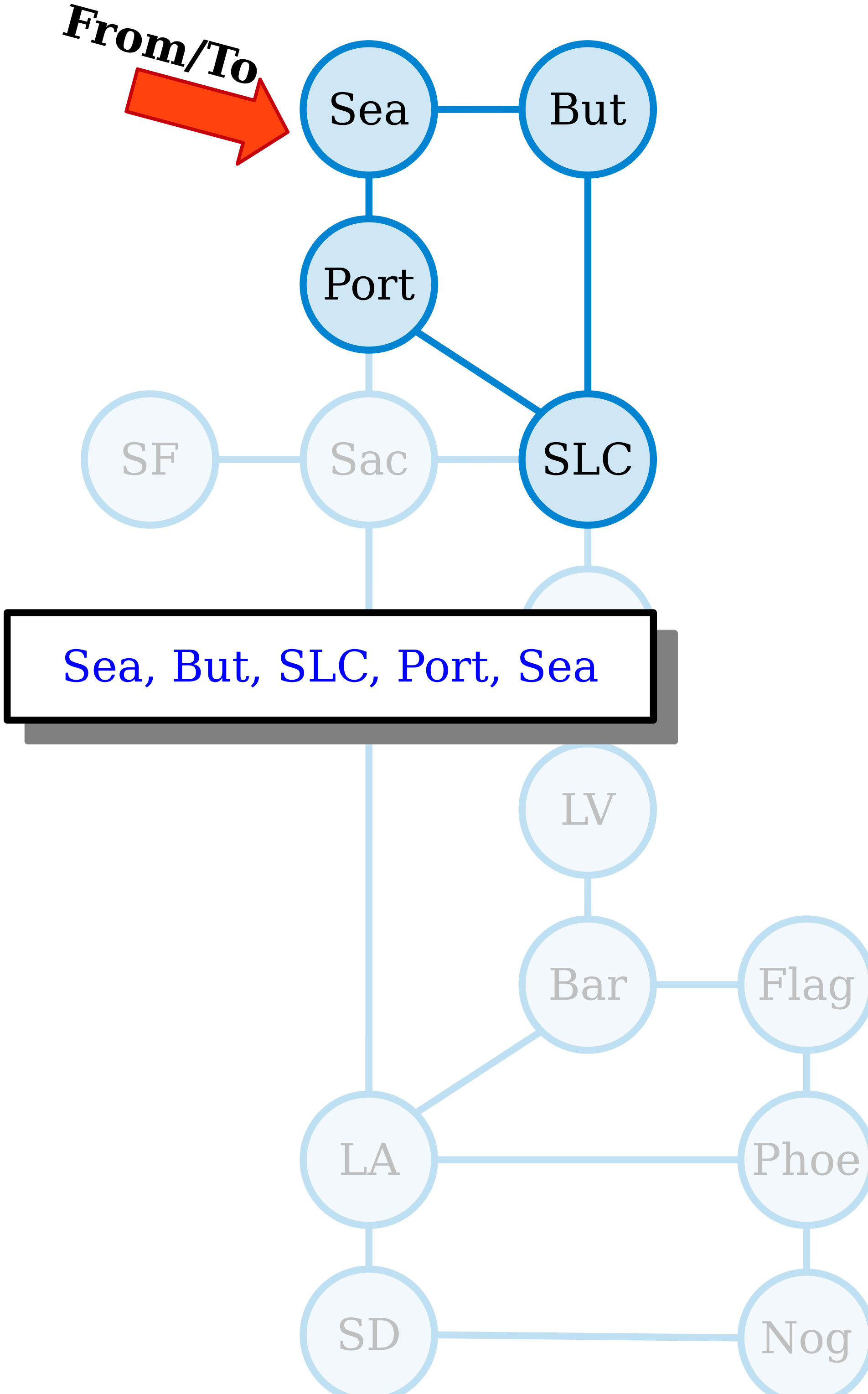

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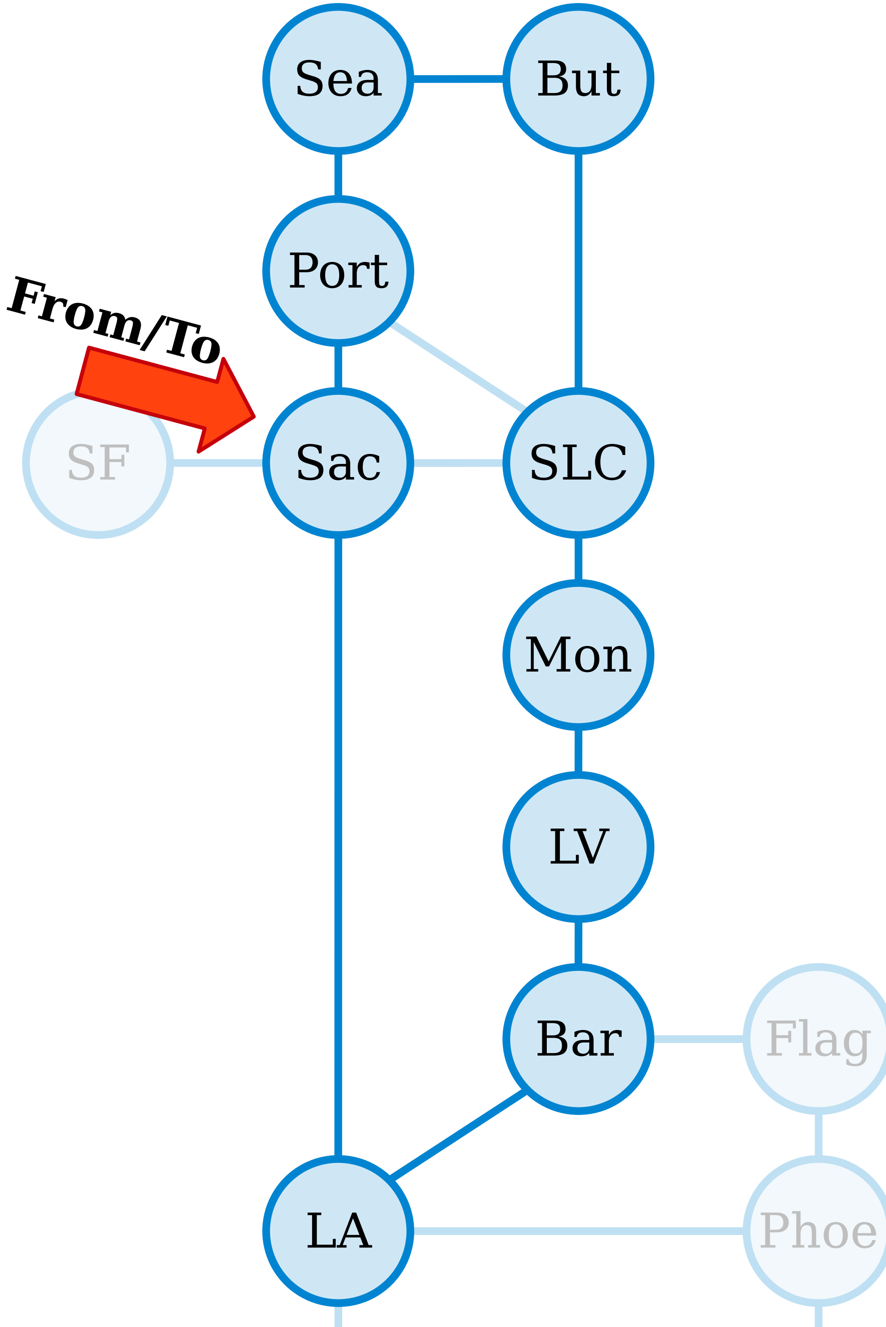

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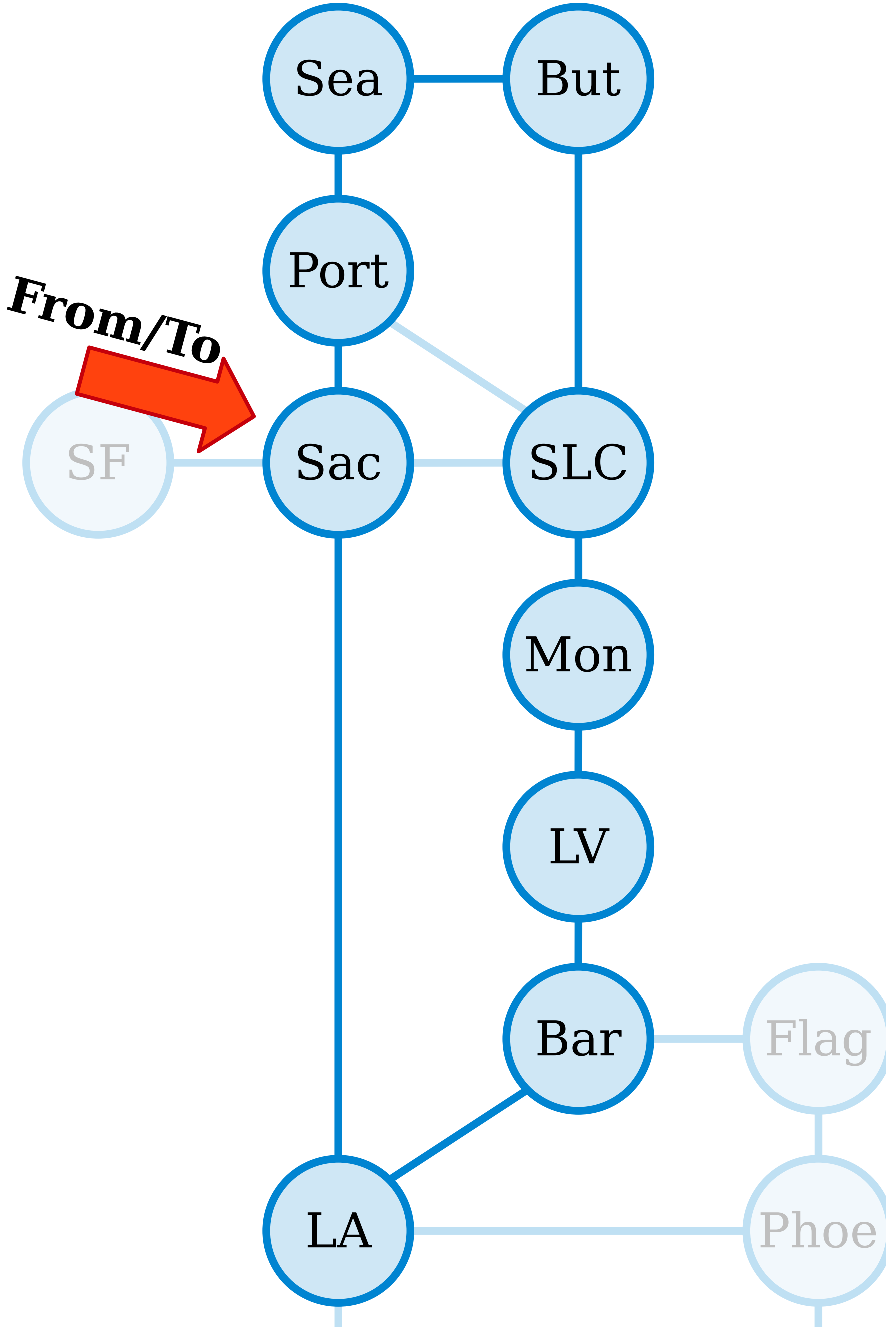
Sac, Port, Sea, But, SLC, Mon, LV, Bar, LA, Sac



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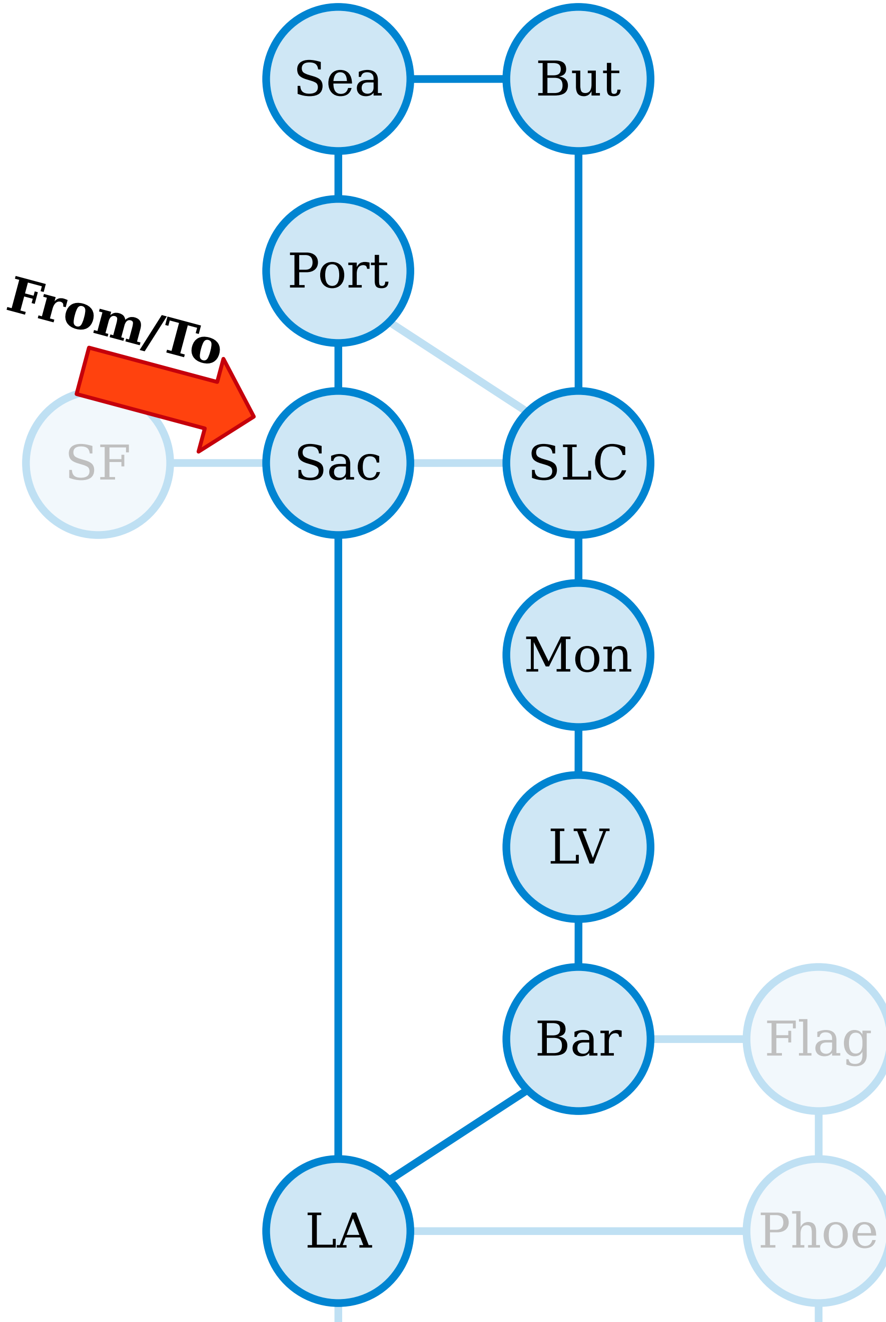

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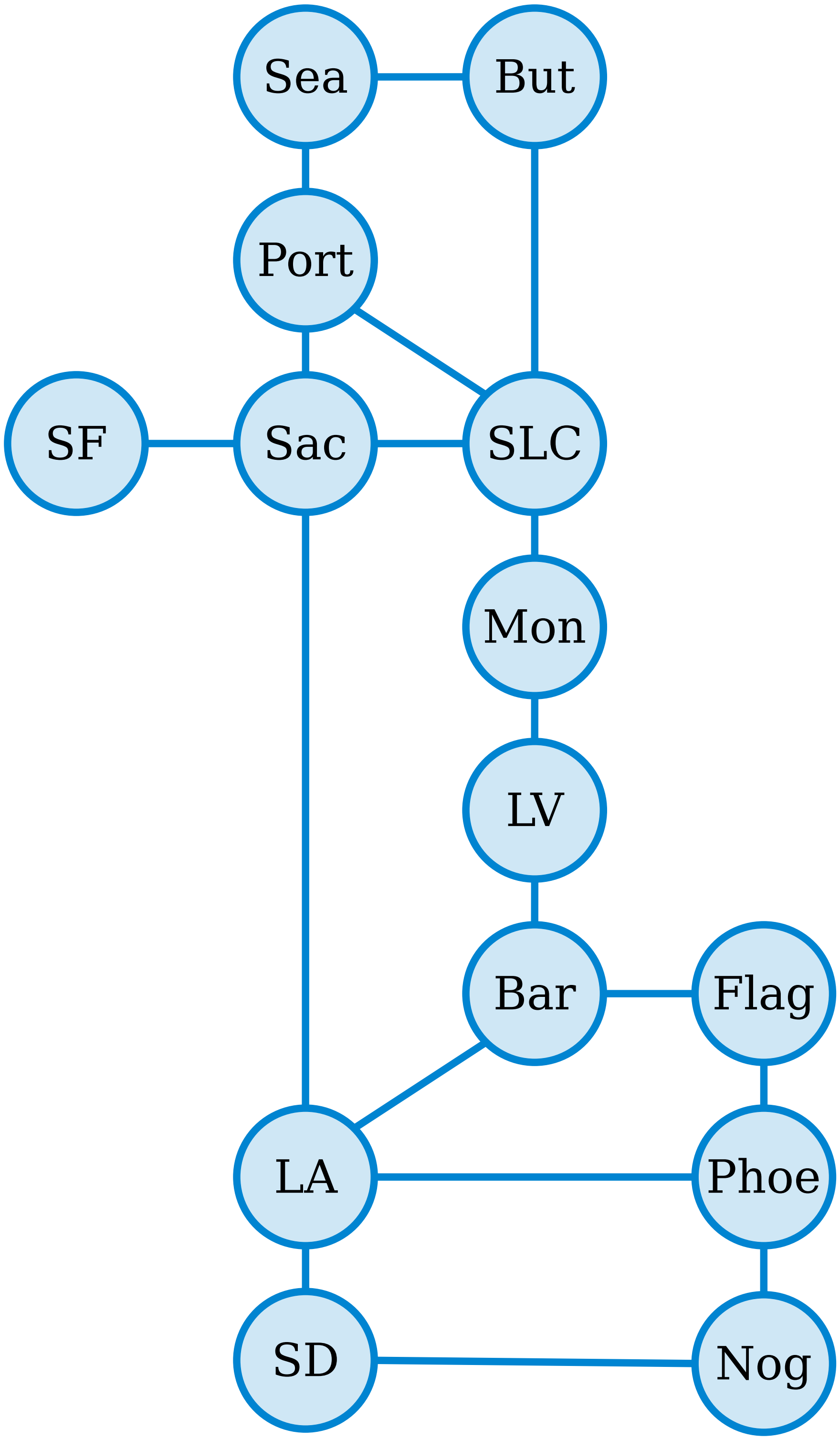
(This closed walk has length nine and visits nine different cities.)

Sac, Port, Sea, But, SLC, Mon, LV, Bar, LA, Sac



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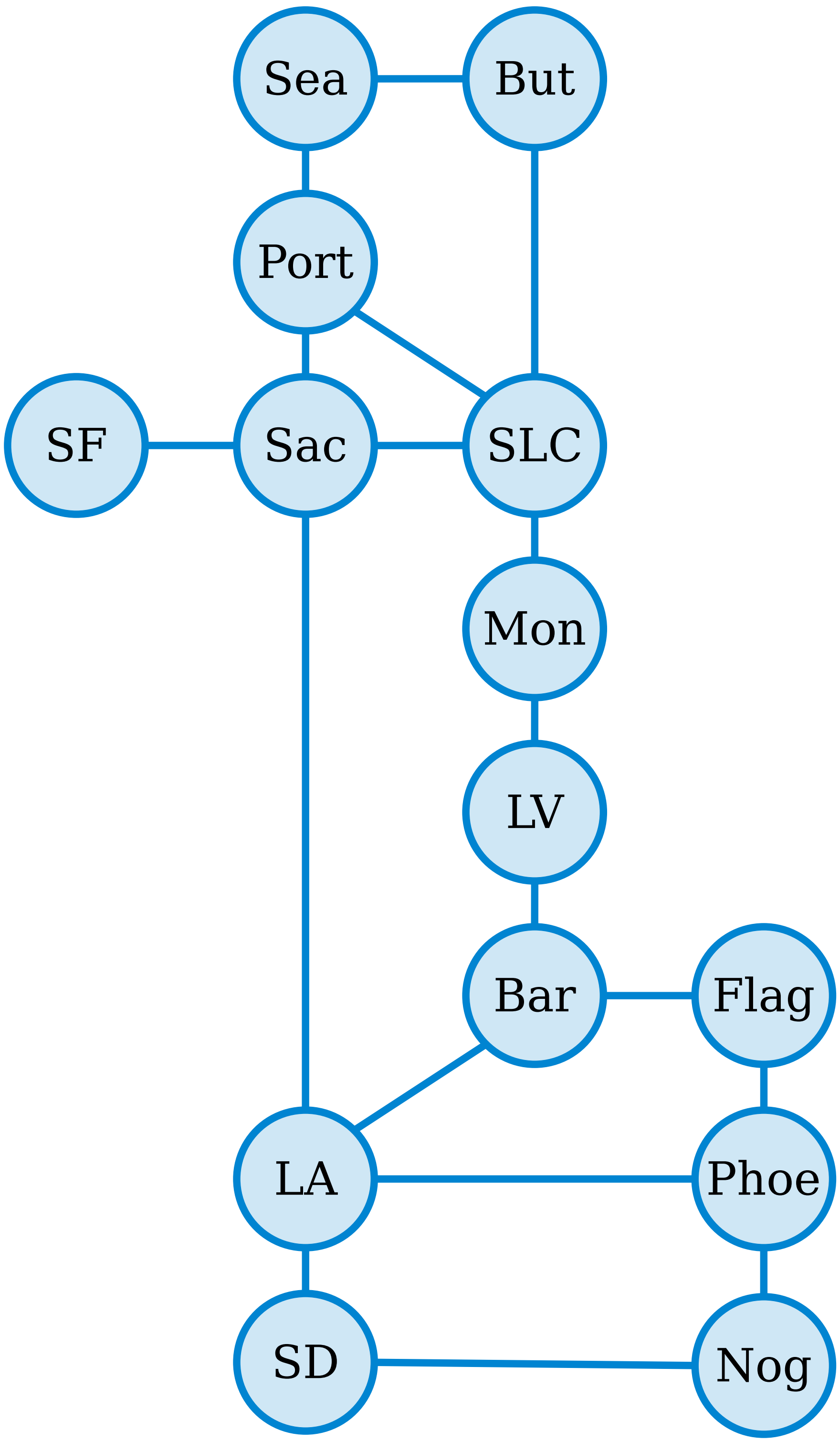

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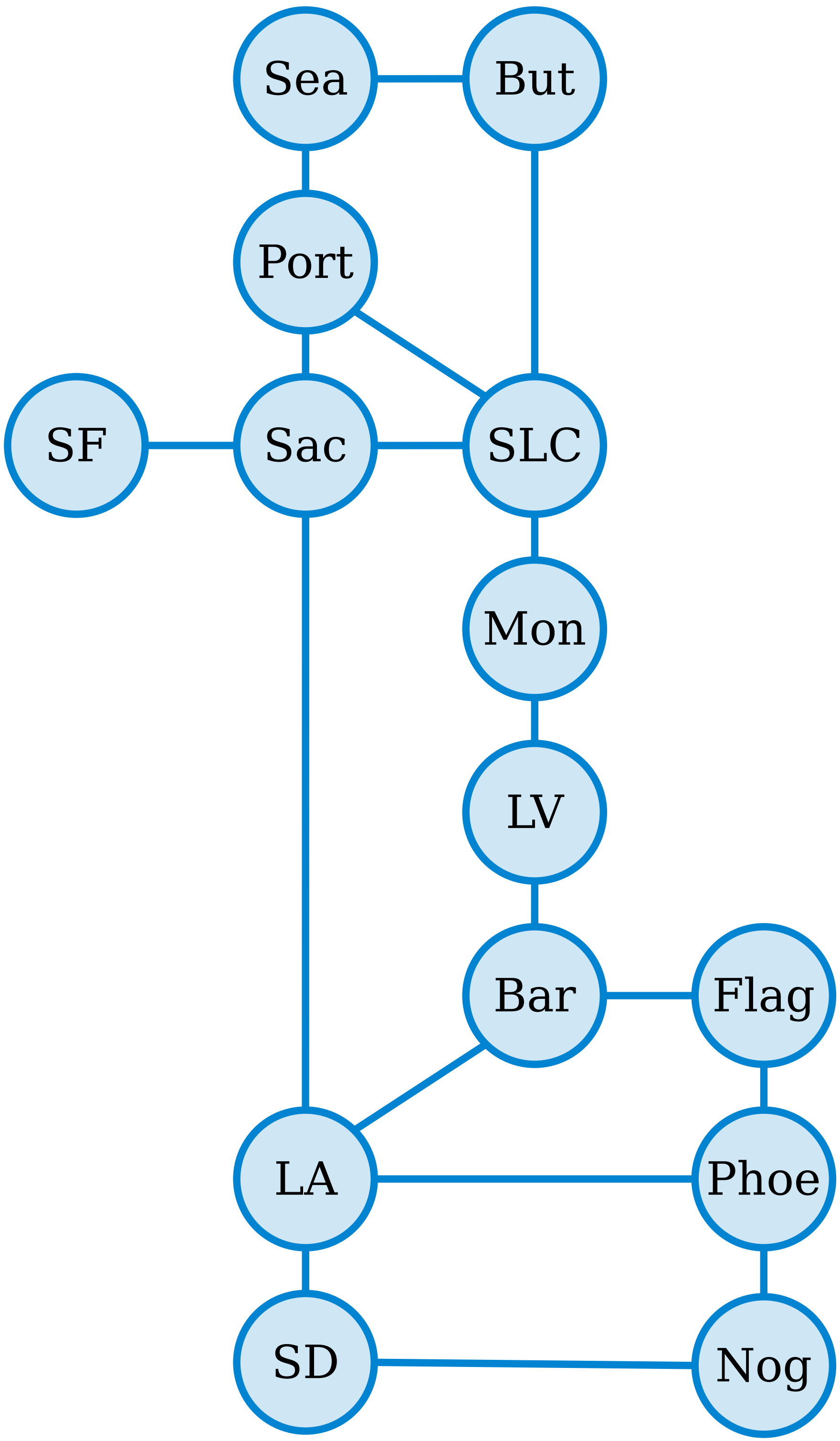

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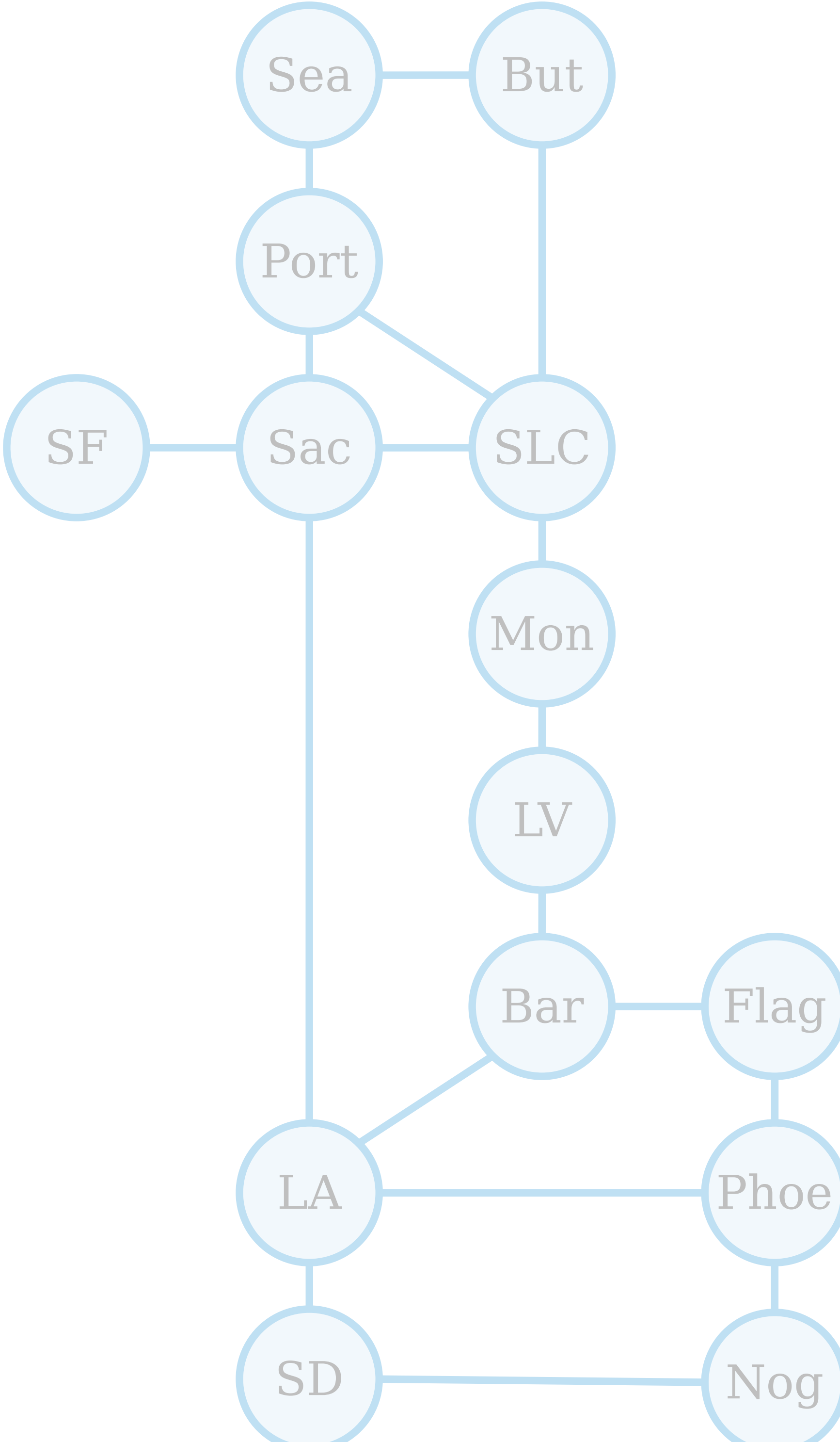

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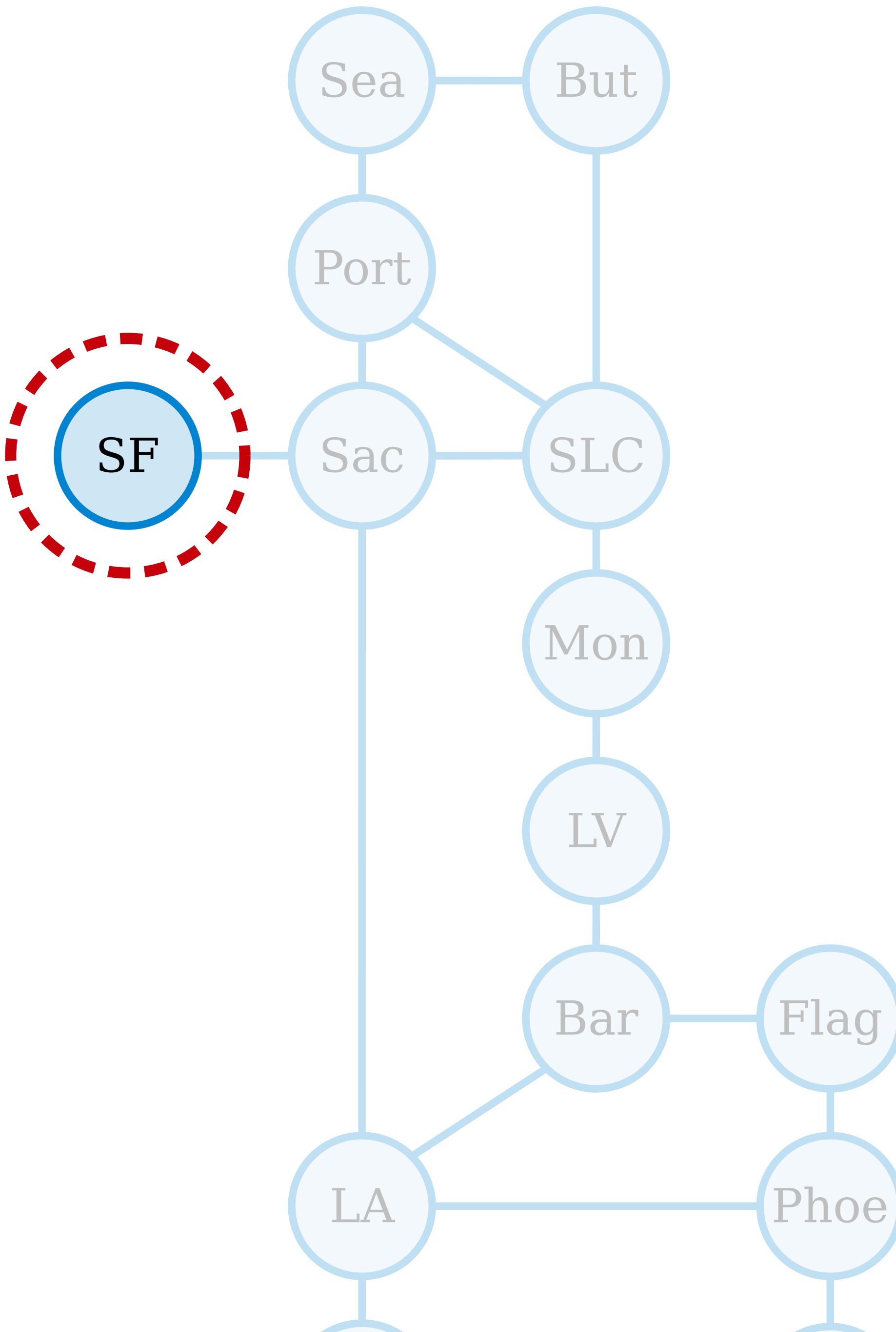

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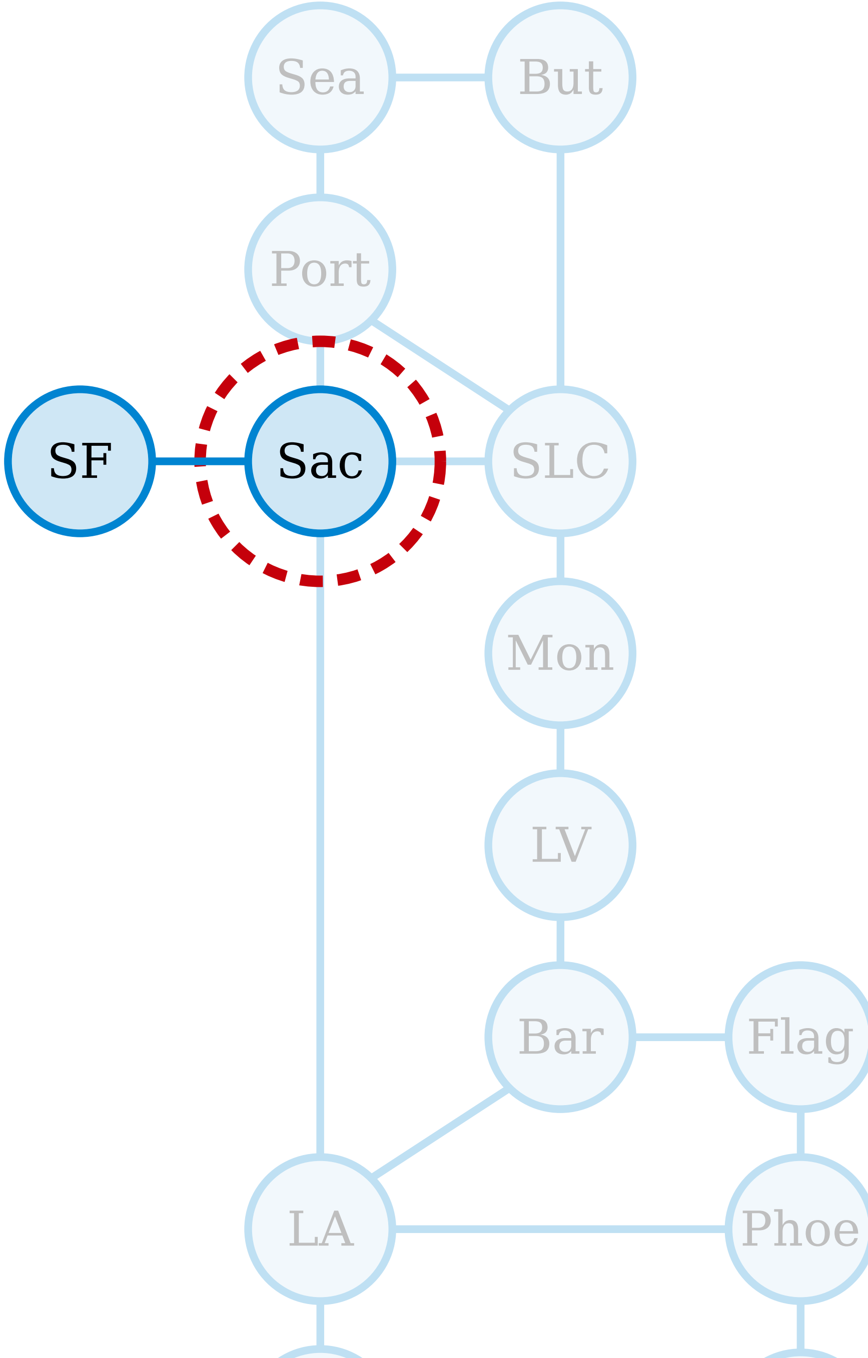

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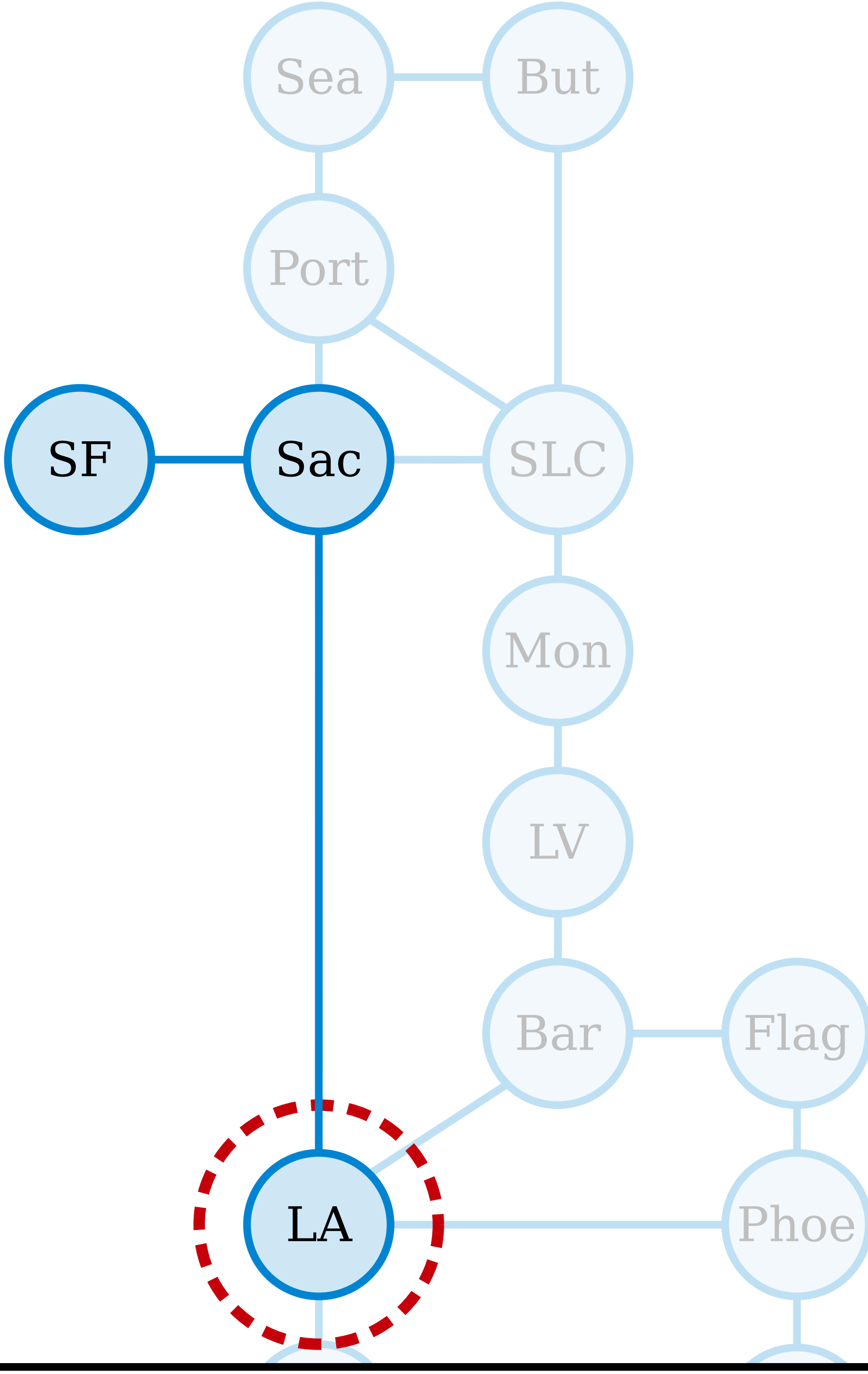
SF, Sac



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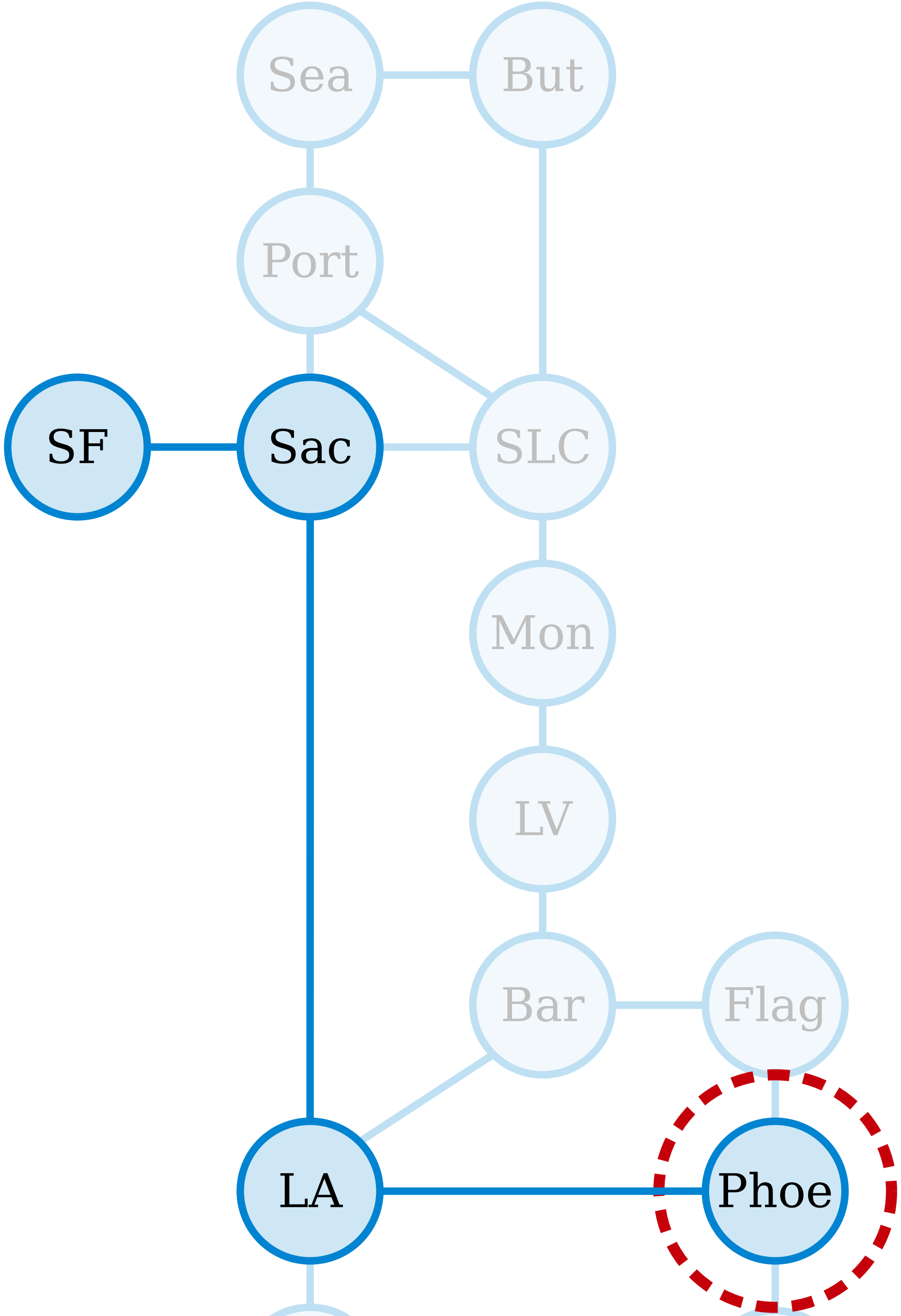
SF, Sac, LA



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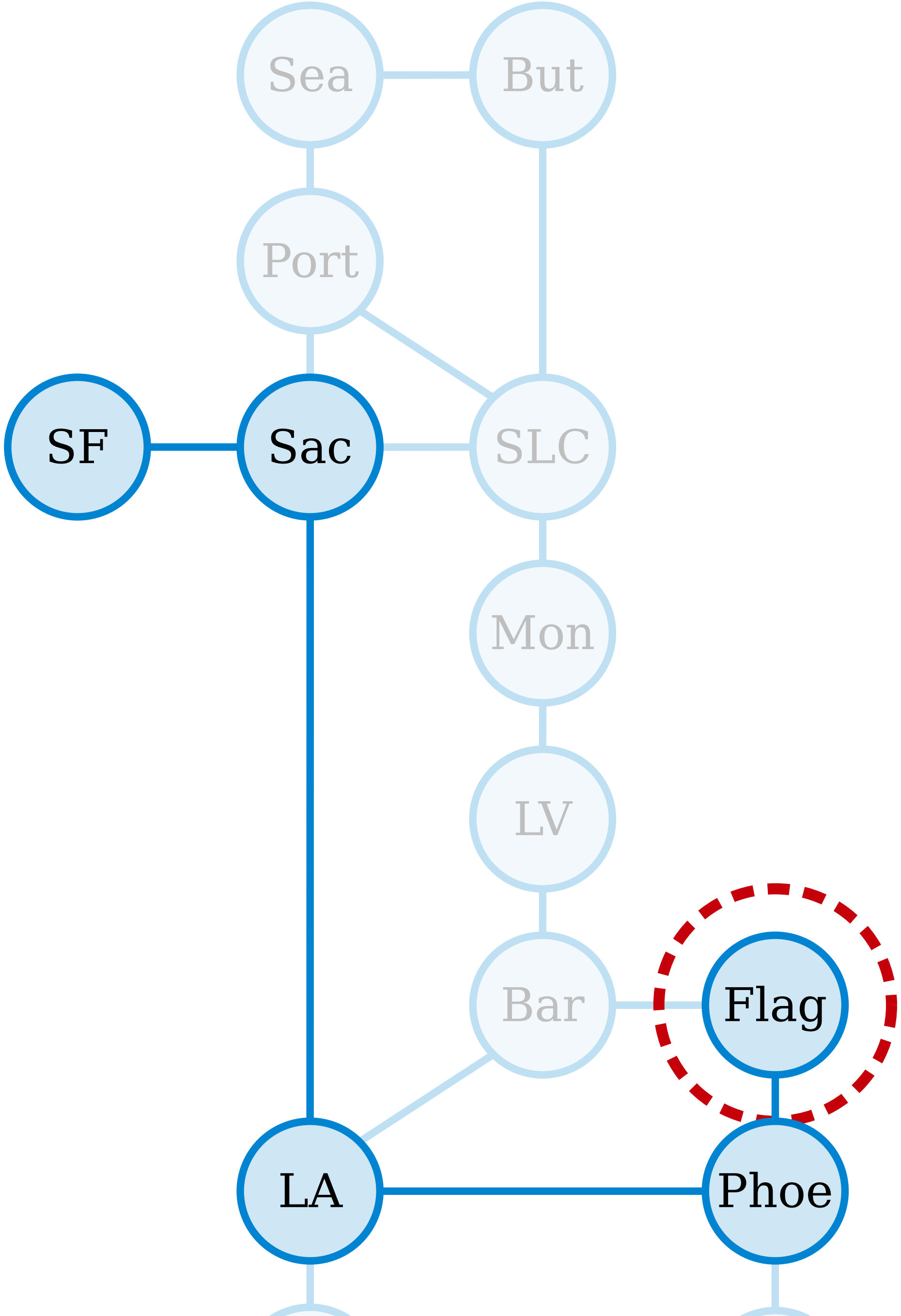
SF, Sac, LA, Phoe



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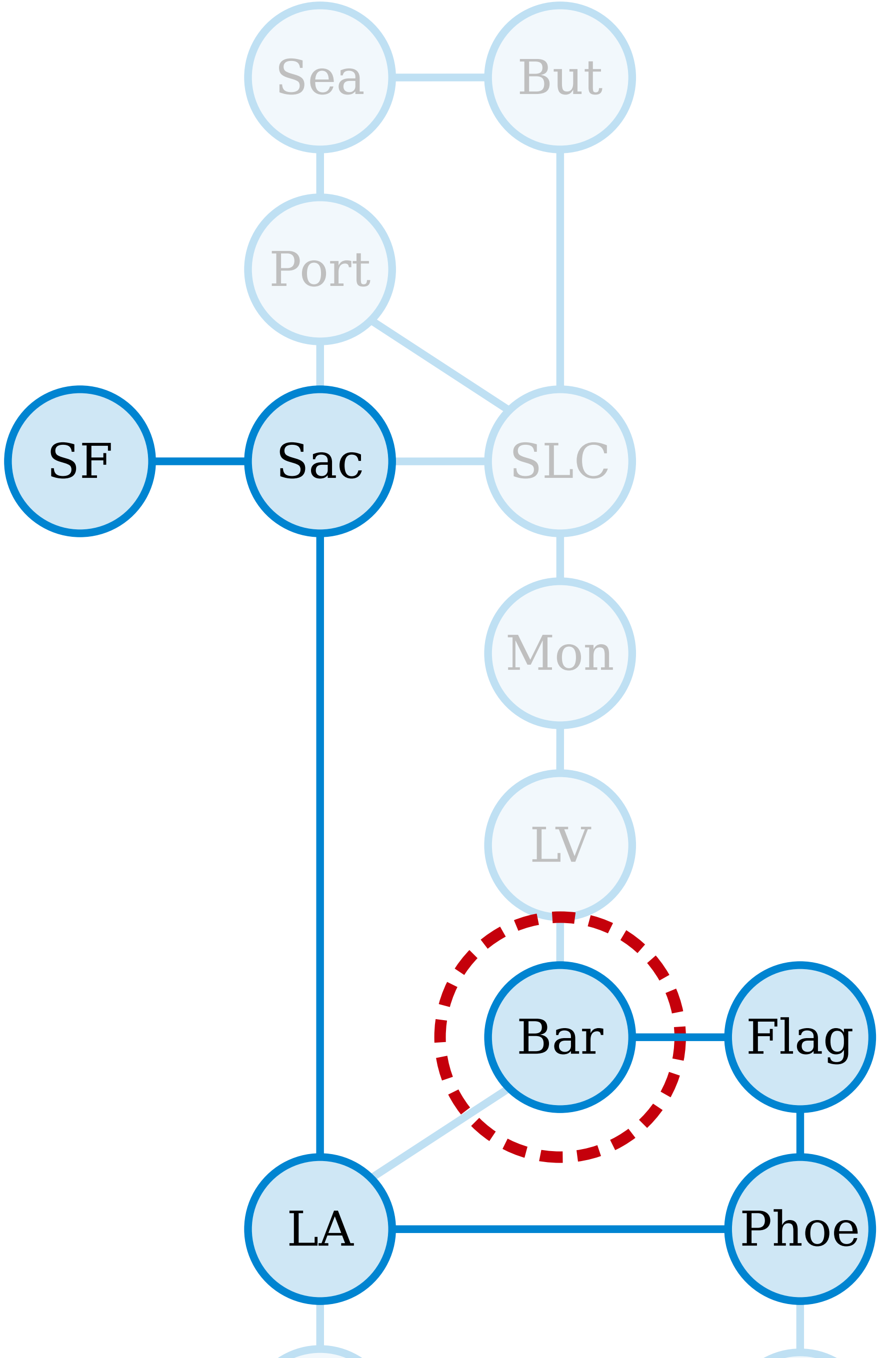
SF, Sac, LA, Phoe, Flag



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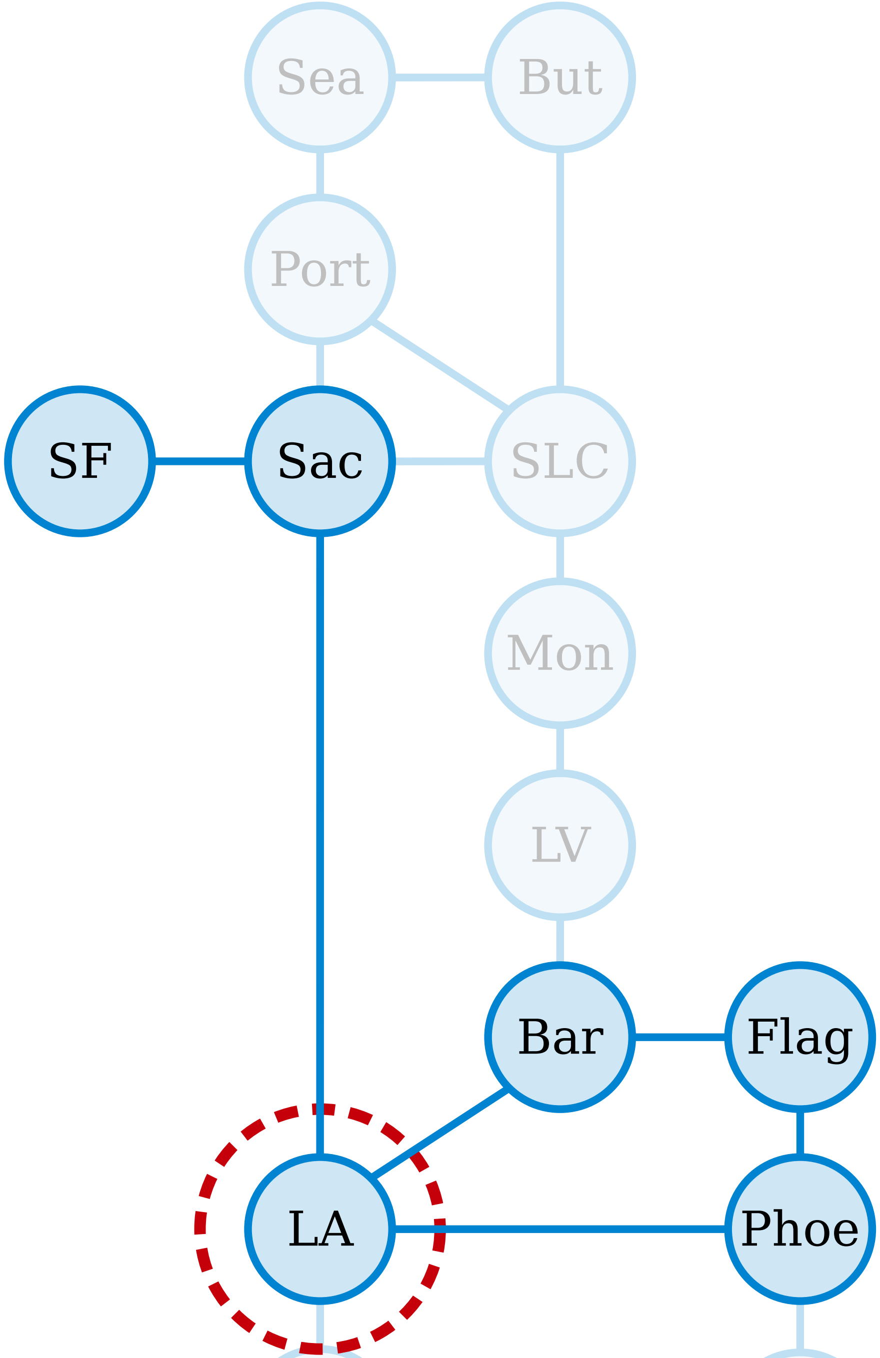

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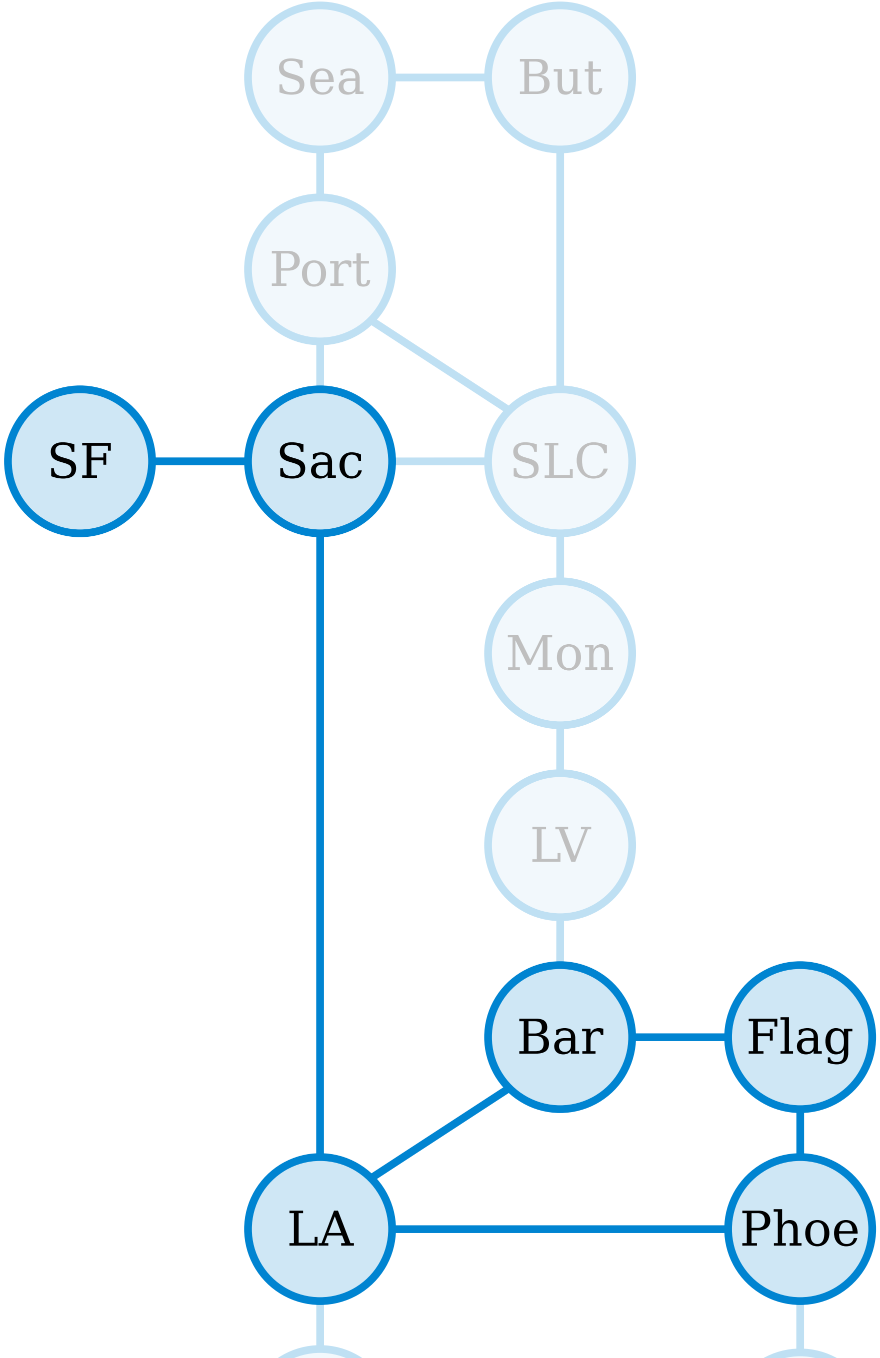

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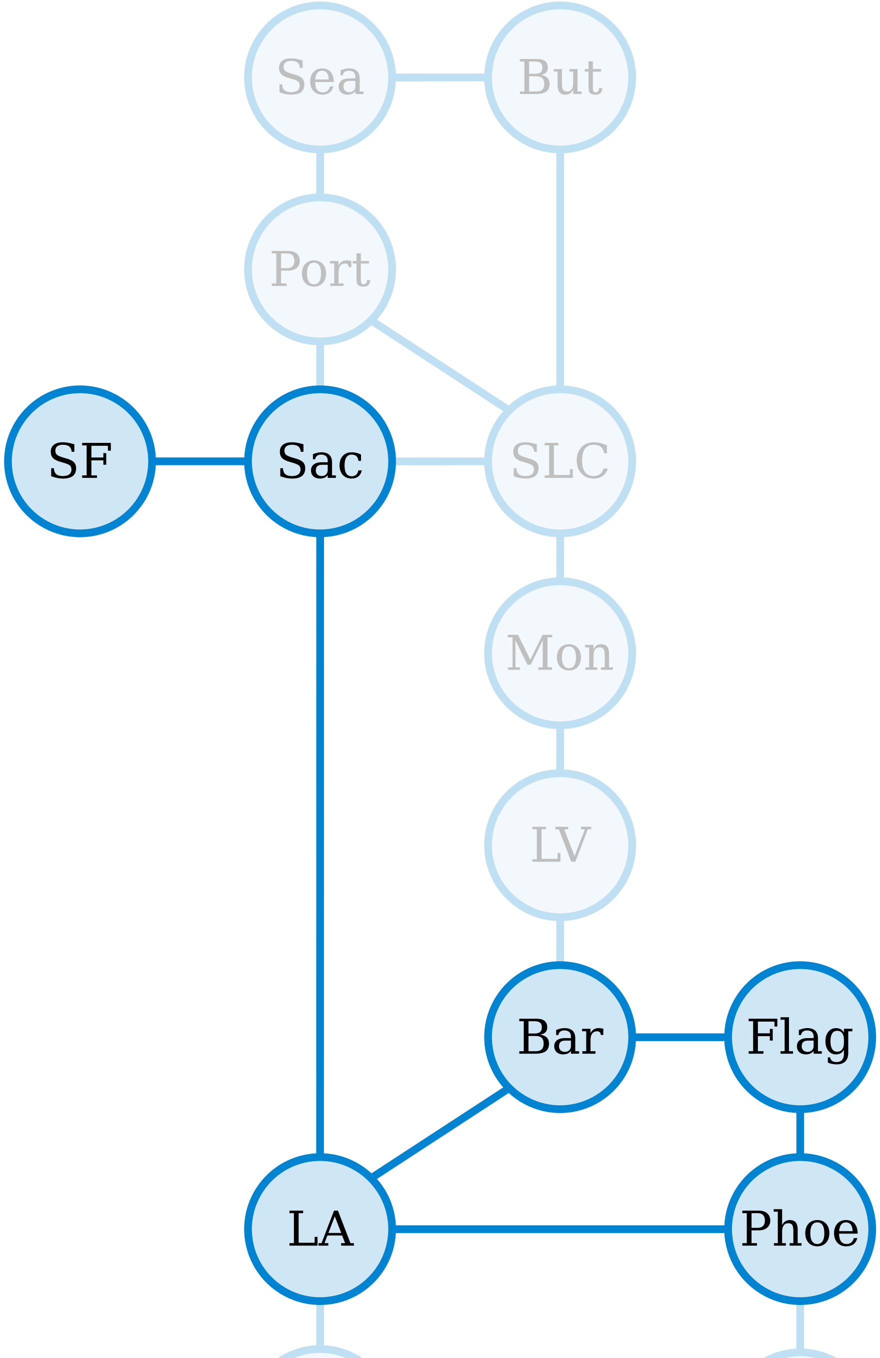

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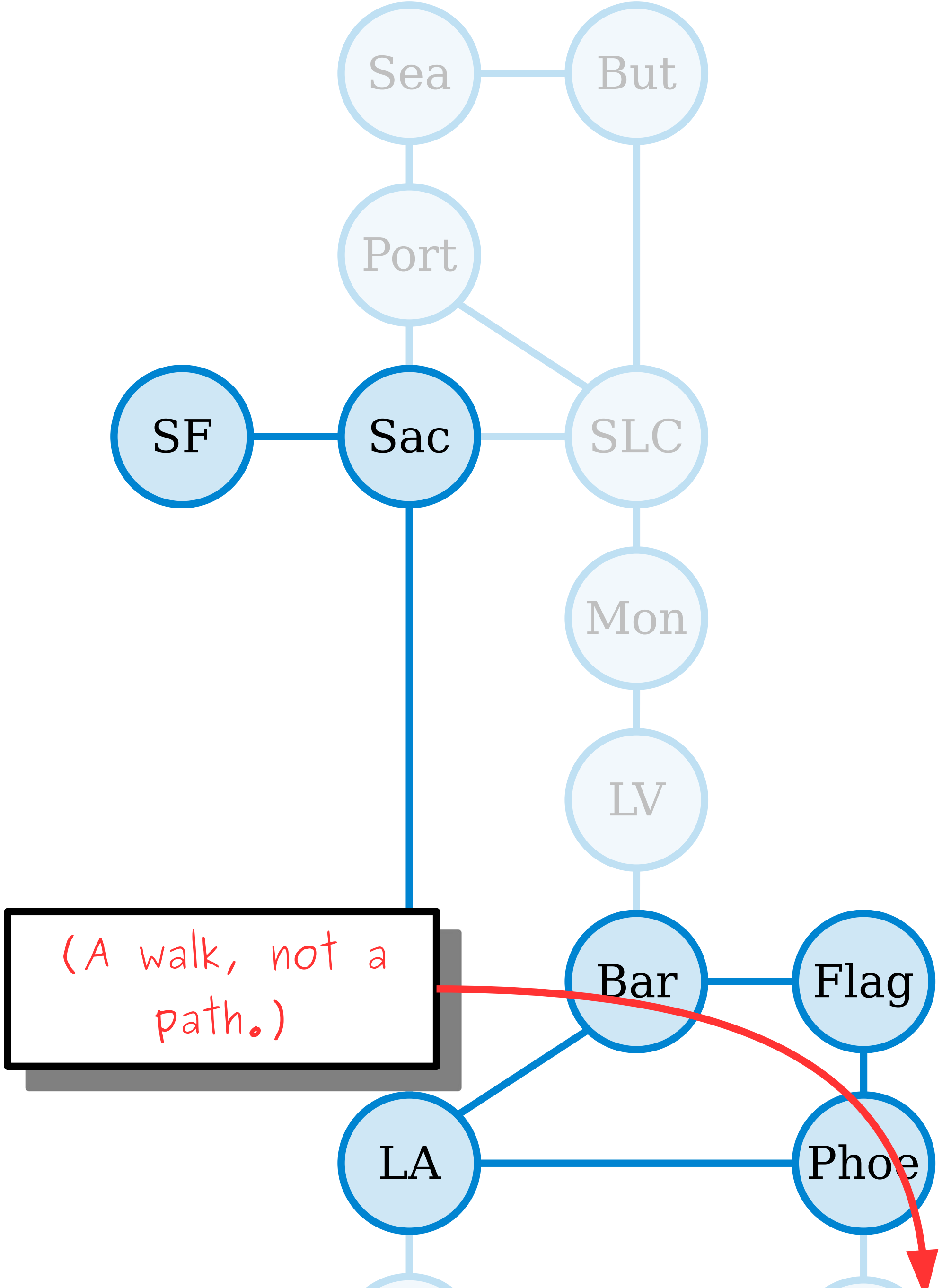
A ***path*** in a graph is walk that does not repeat any nodes.

SF, Sac, LA, Phoe, Flag, Bar, LA



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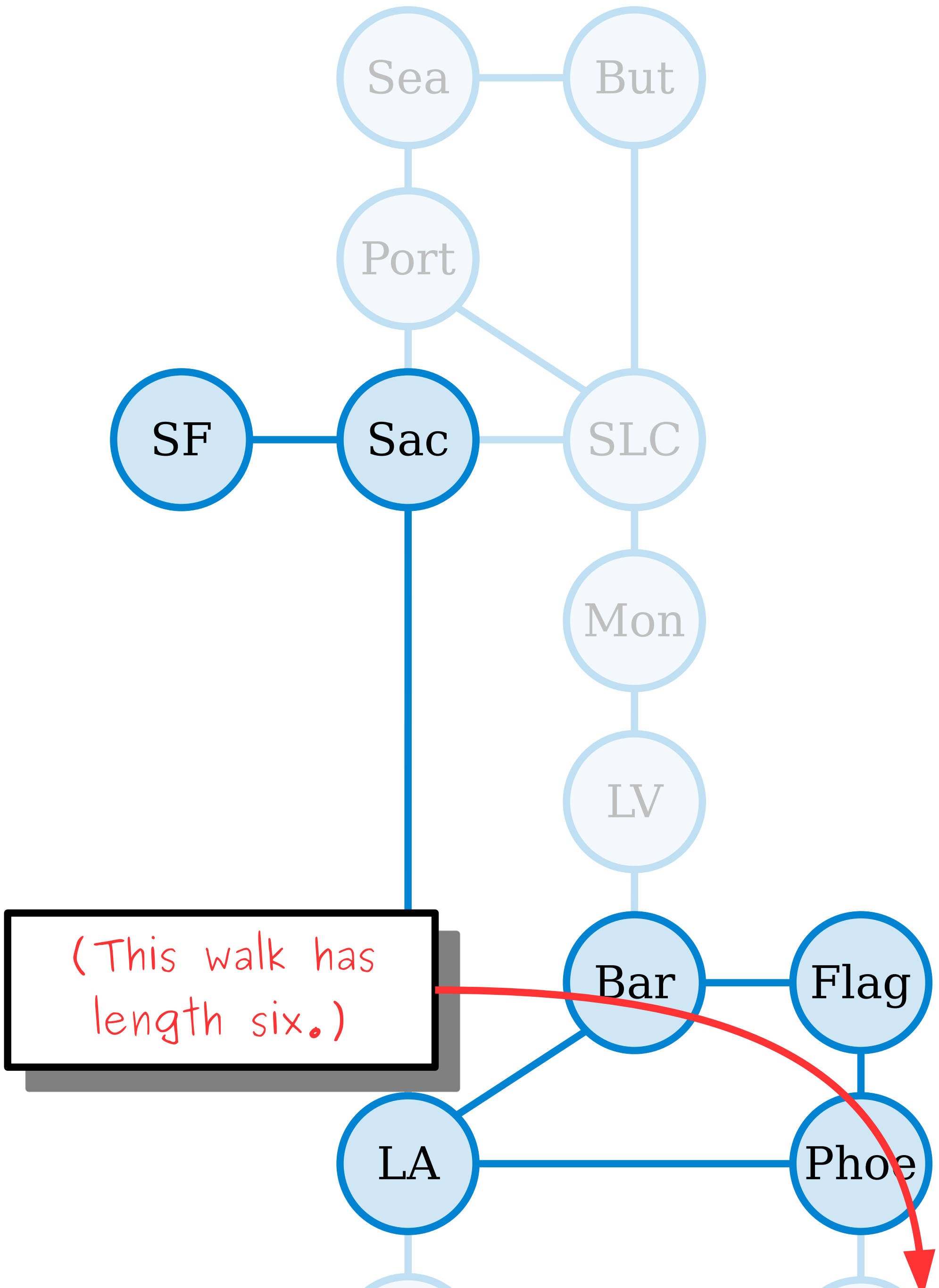

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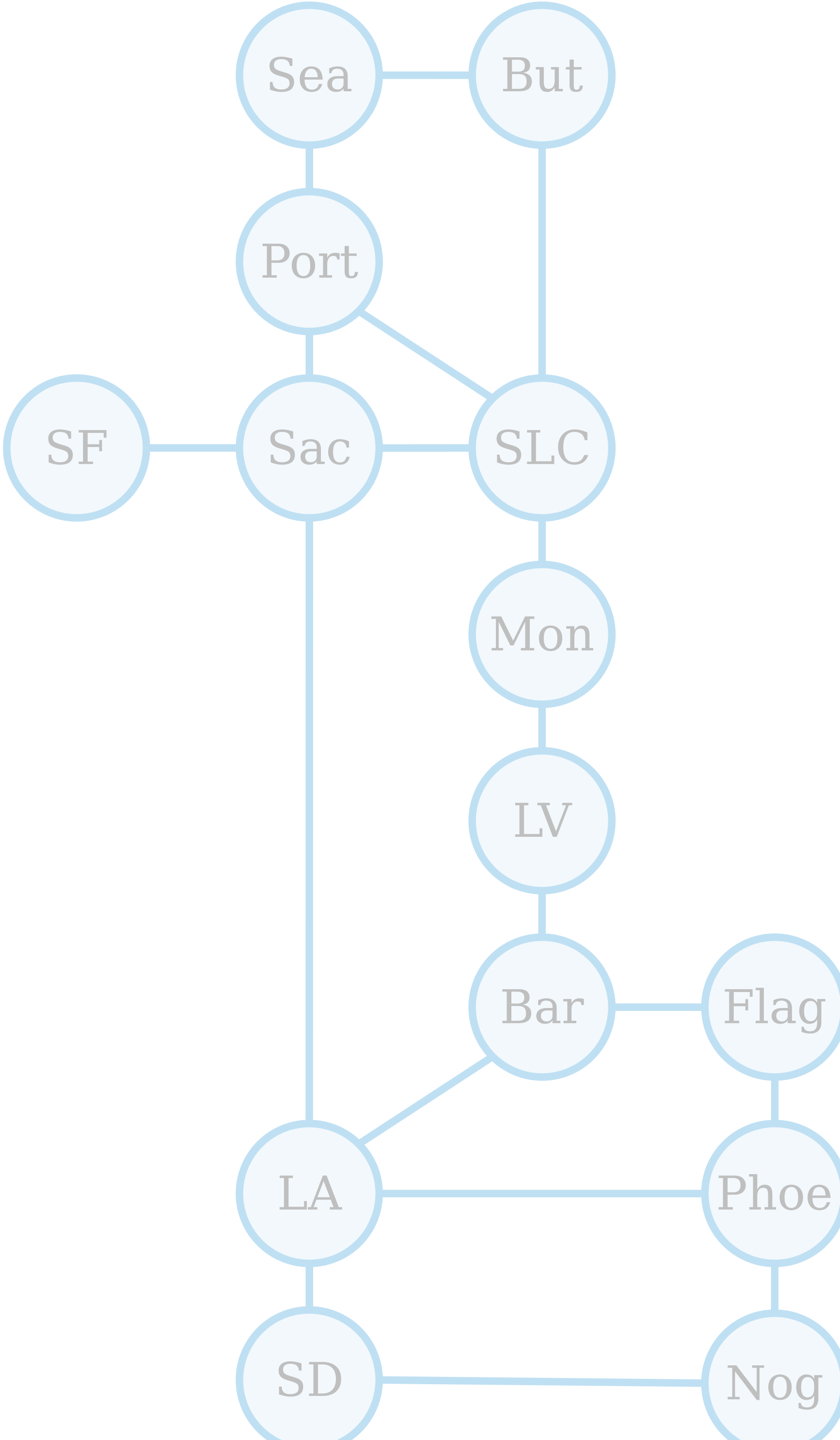

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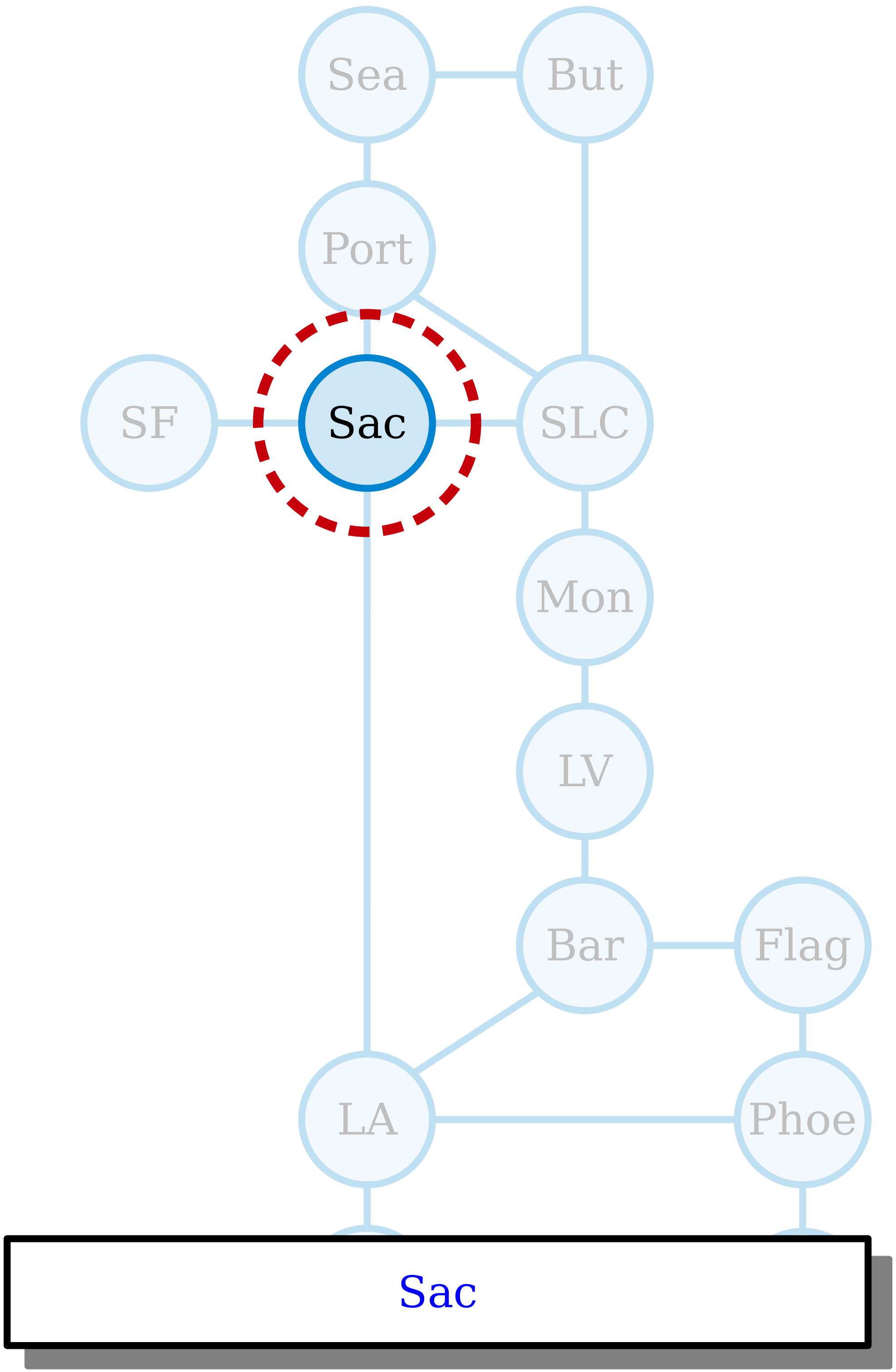

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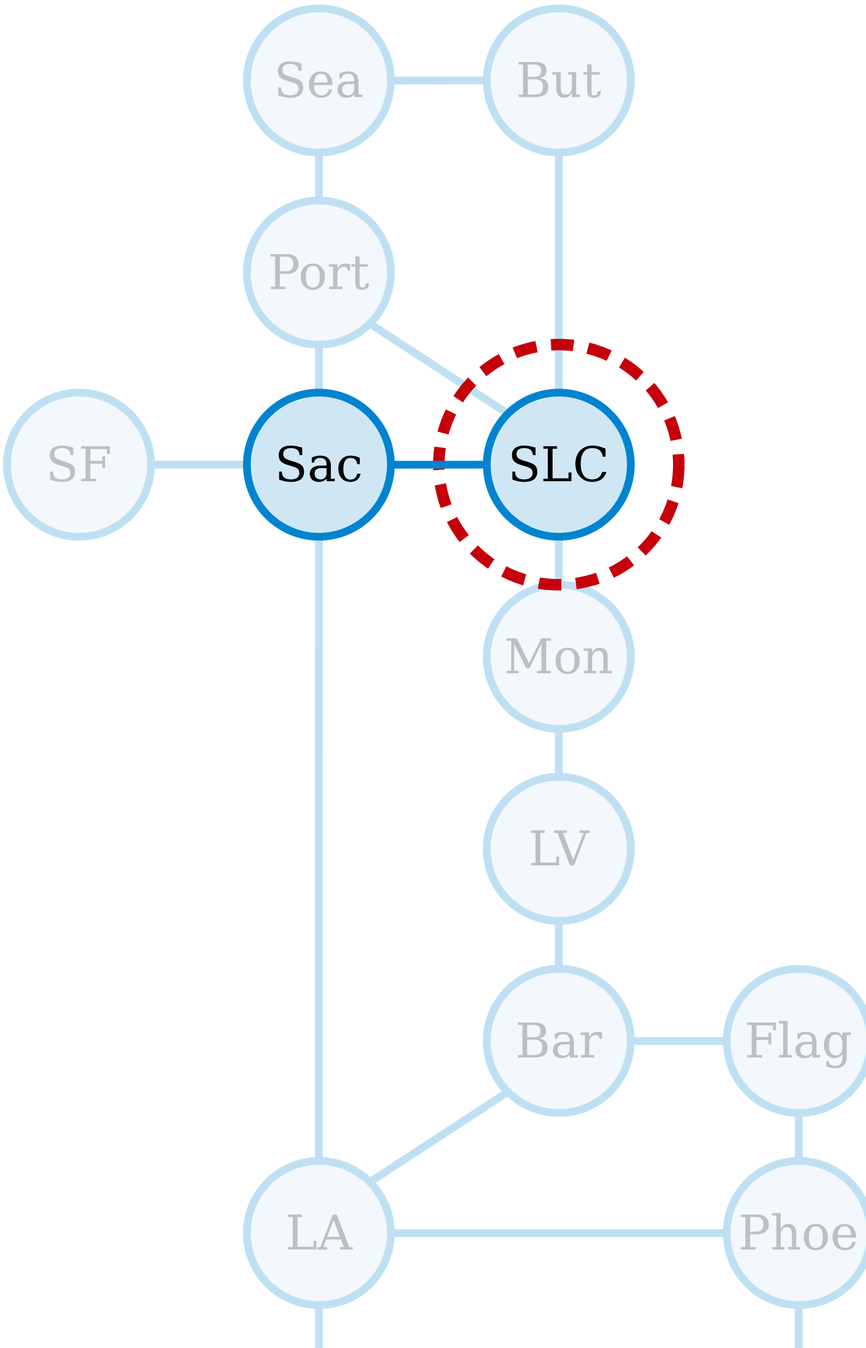

Sac, SLC

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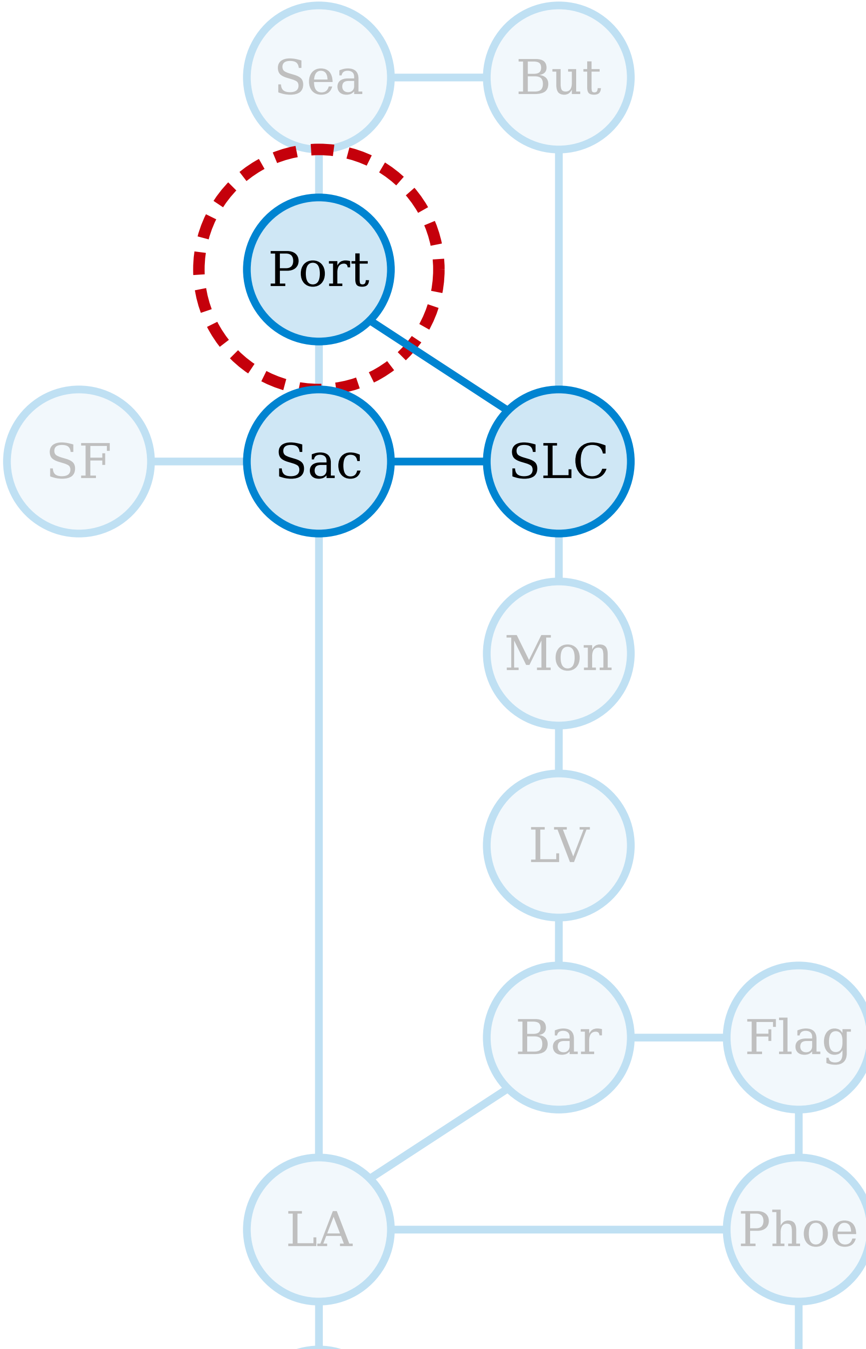

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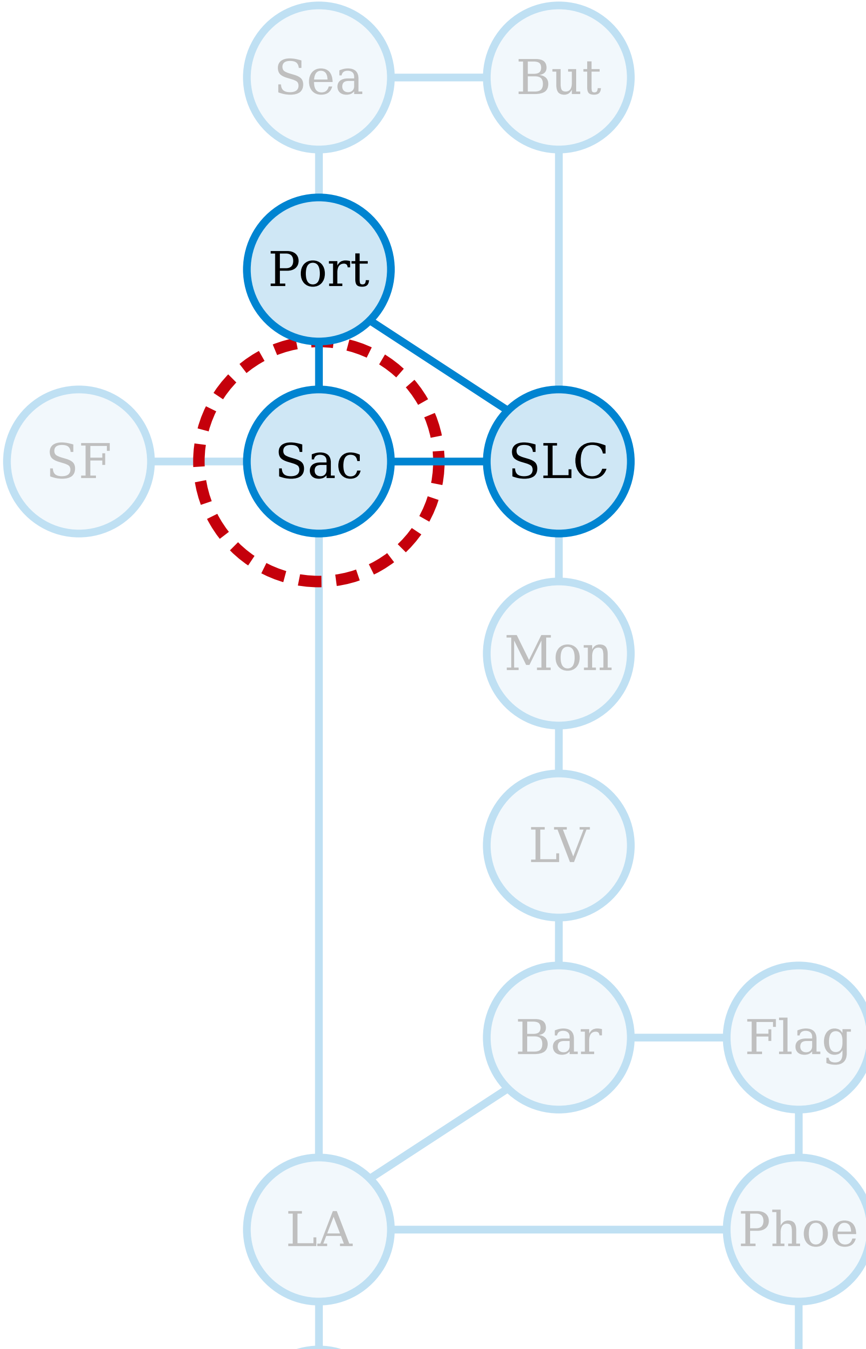

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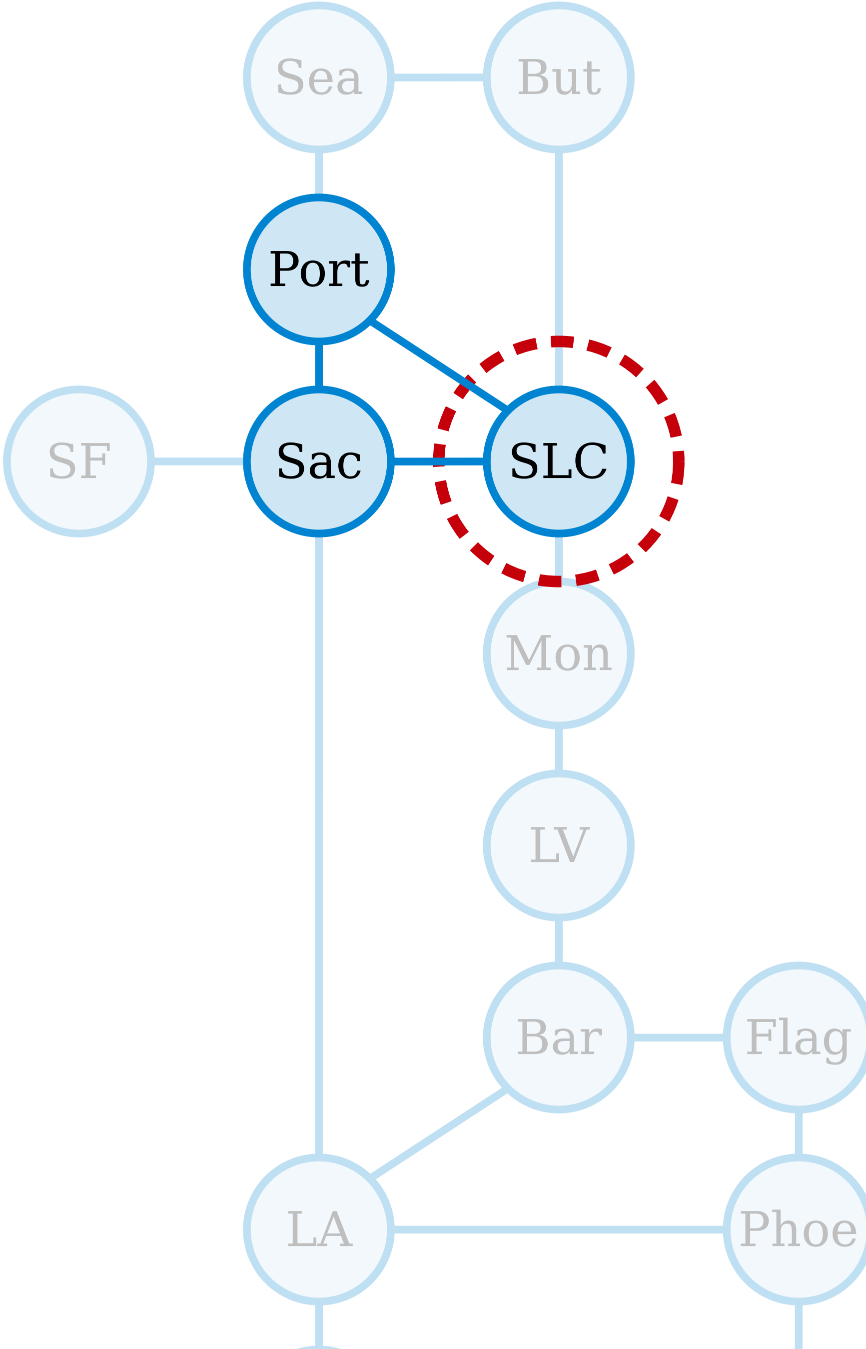

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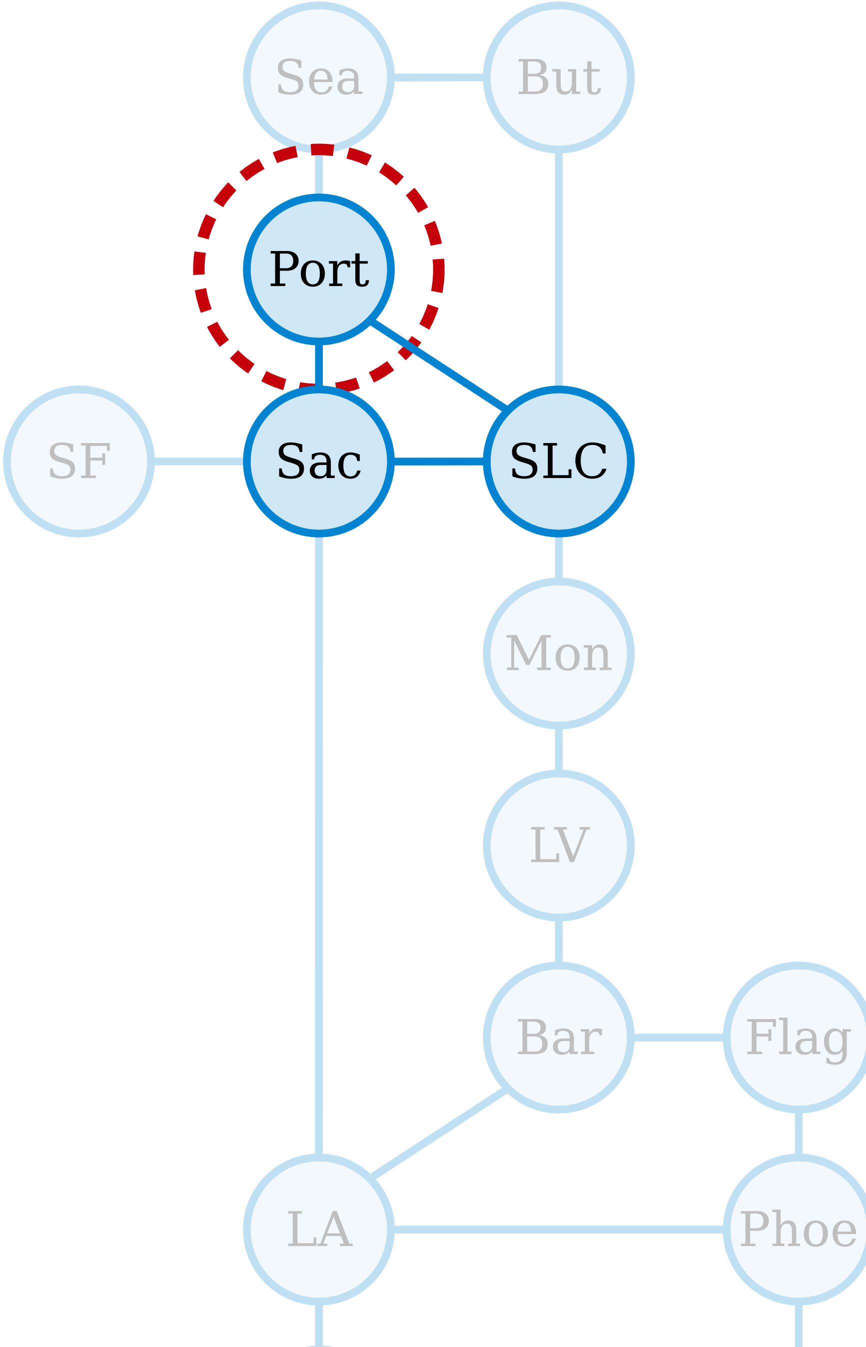

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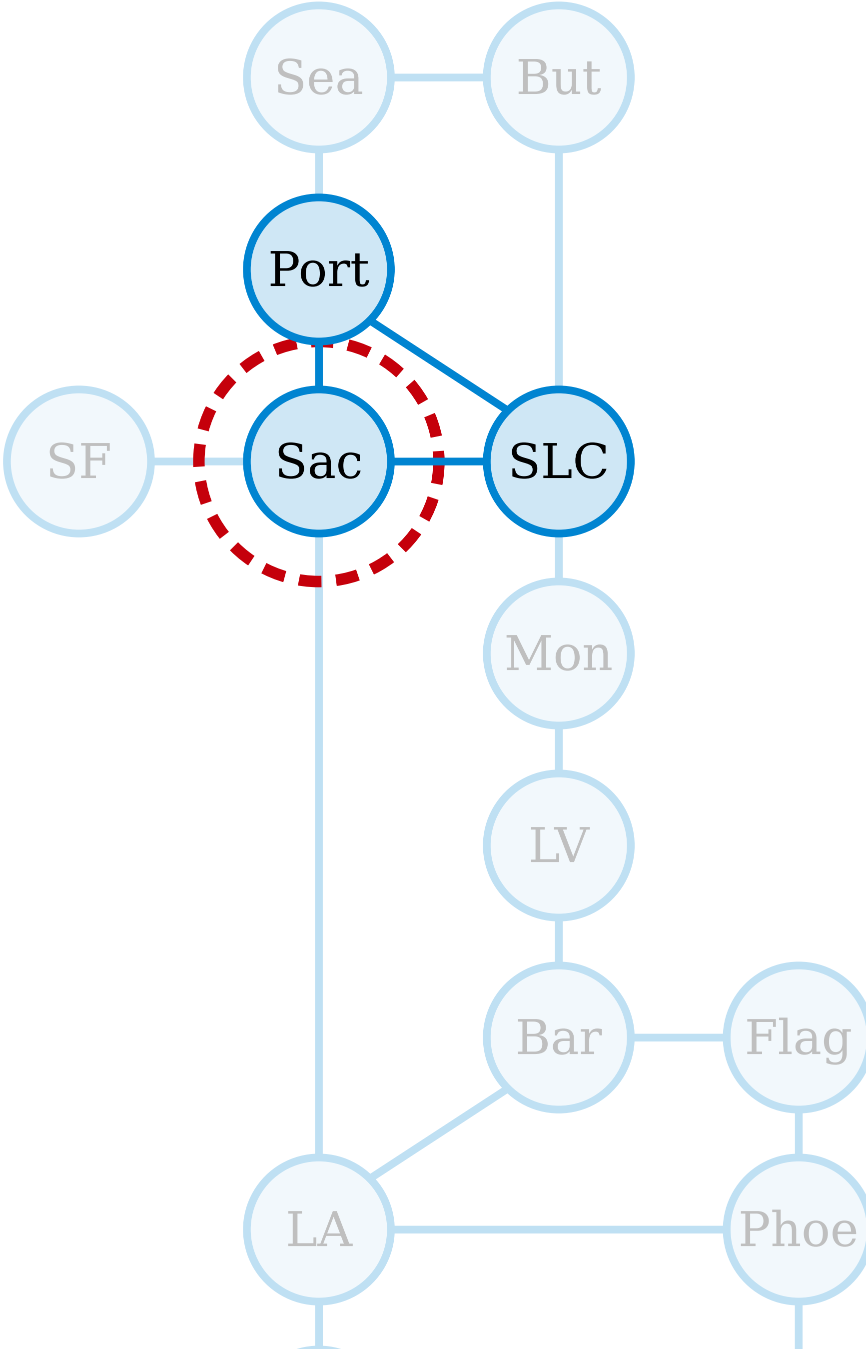

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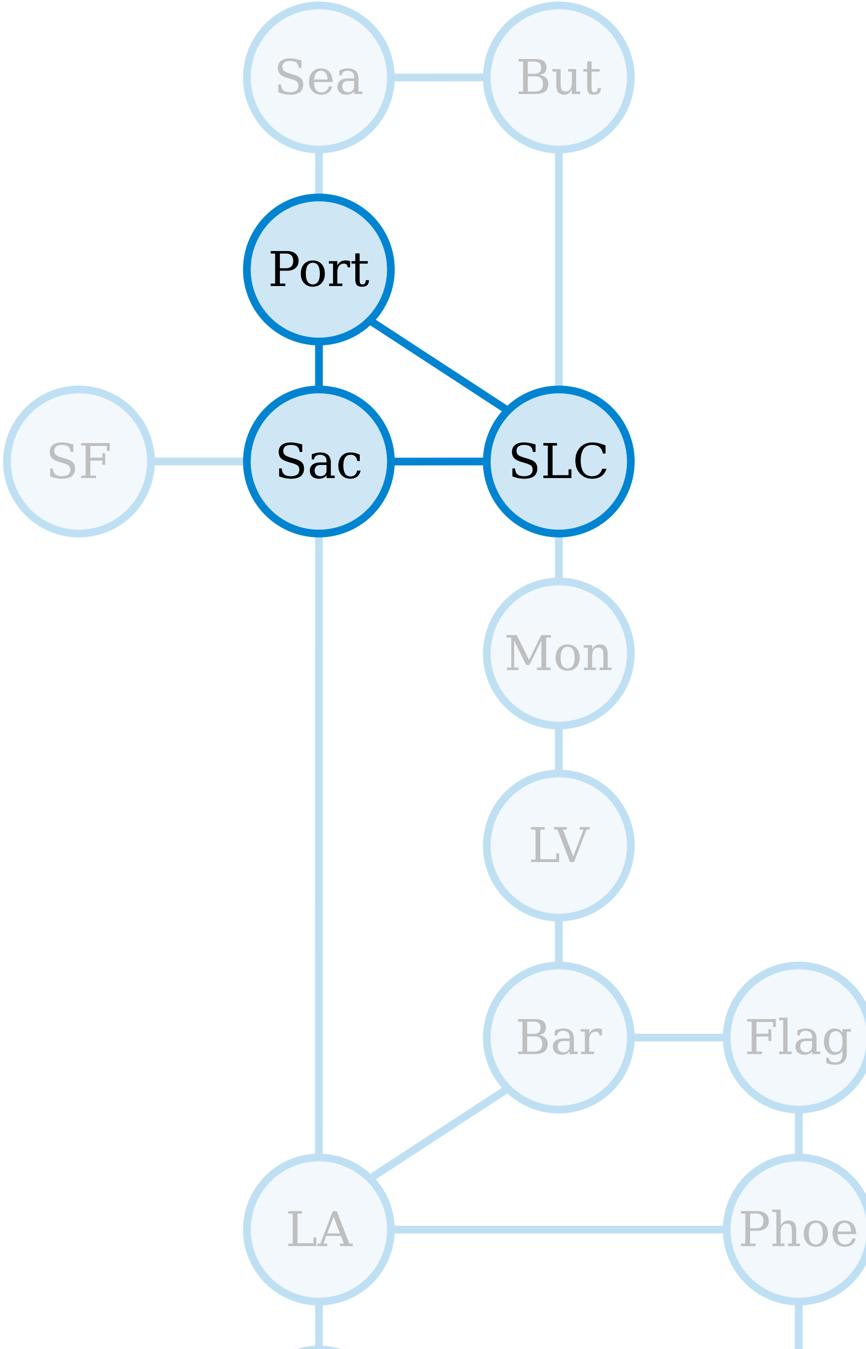

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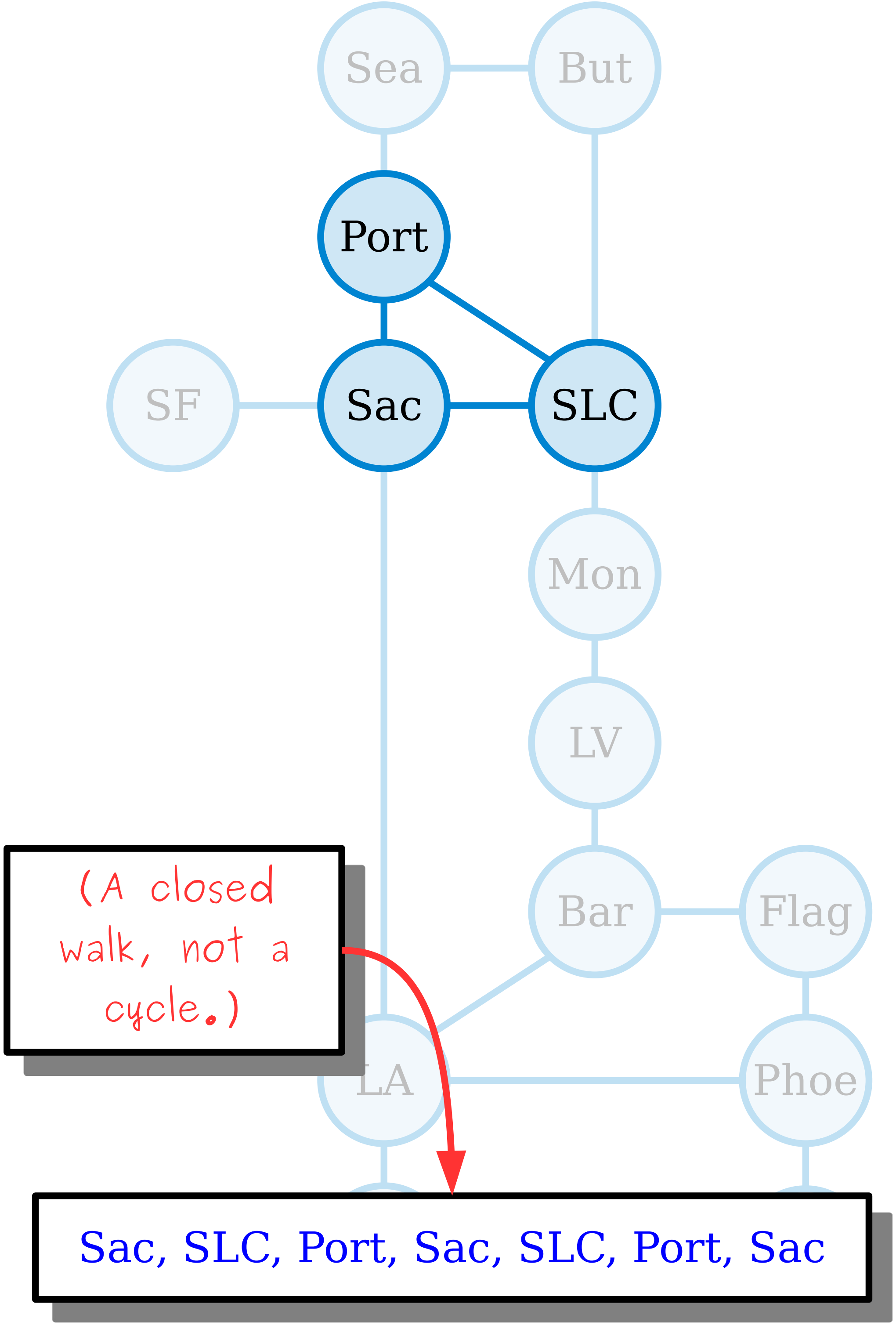

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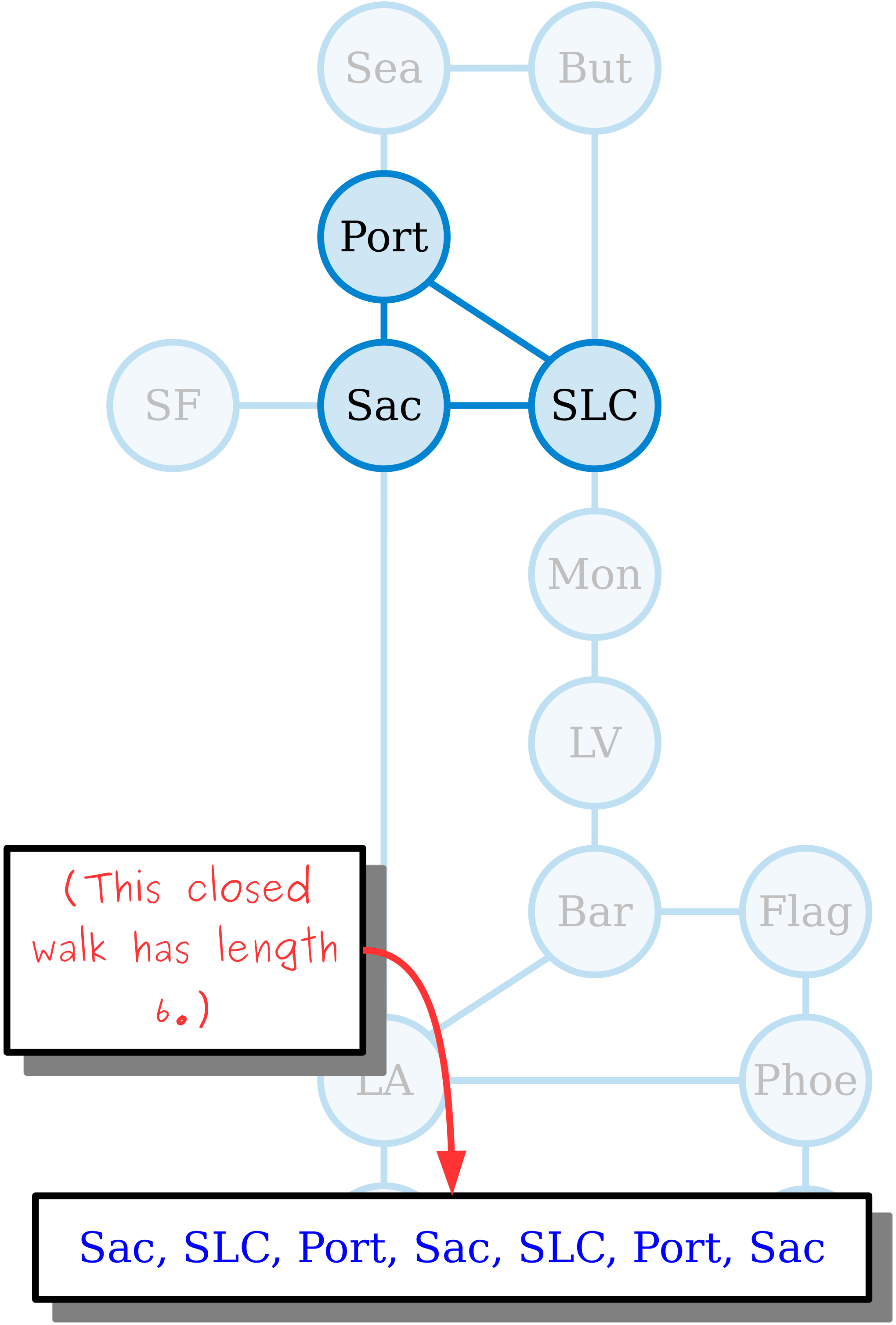

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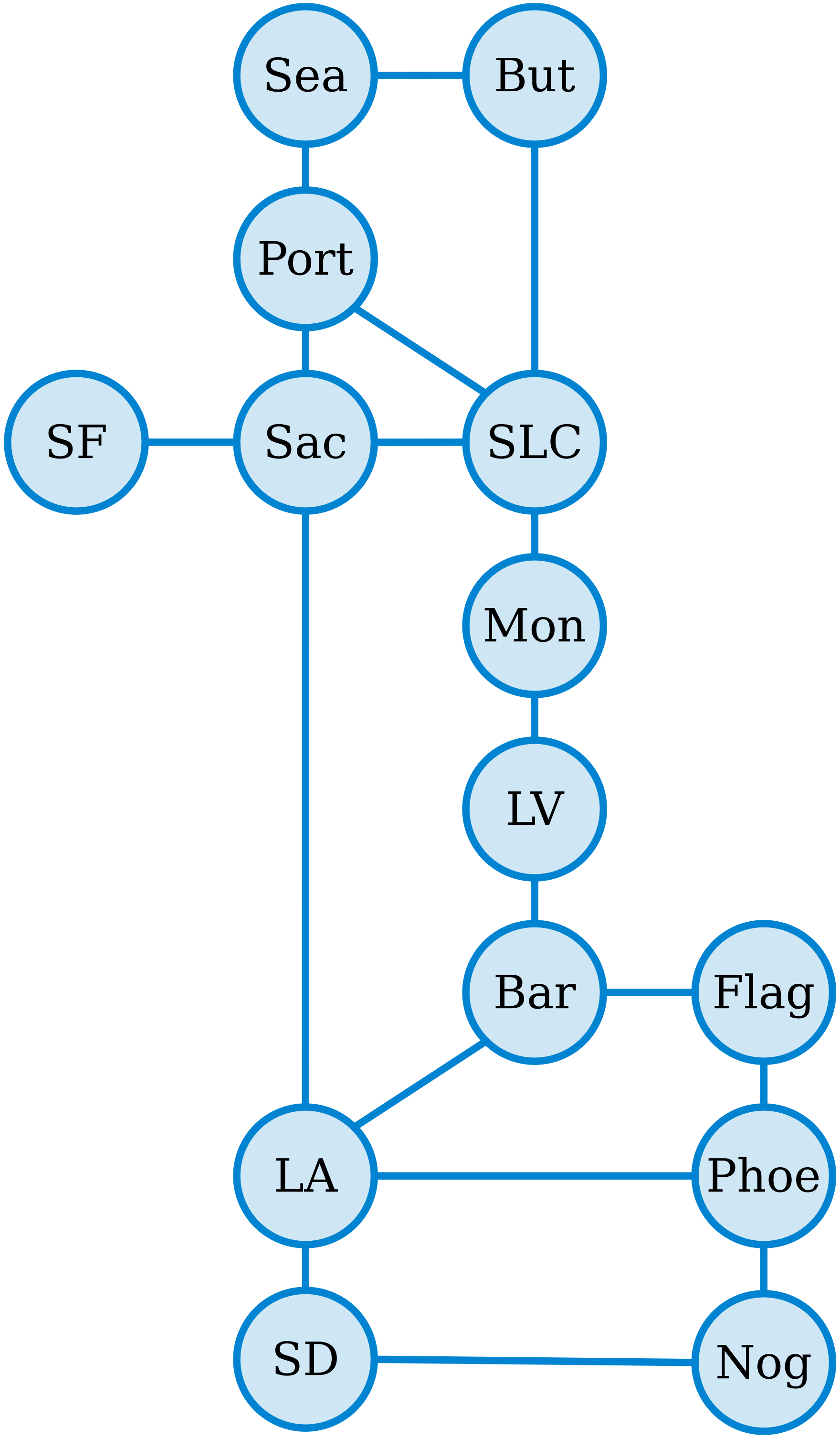

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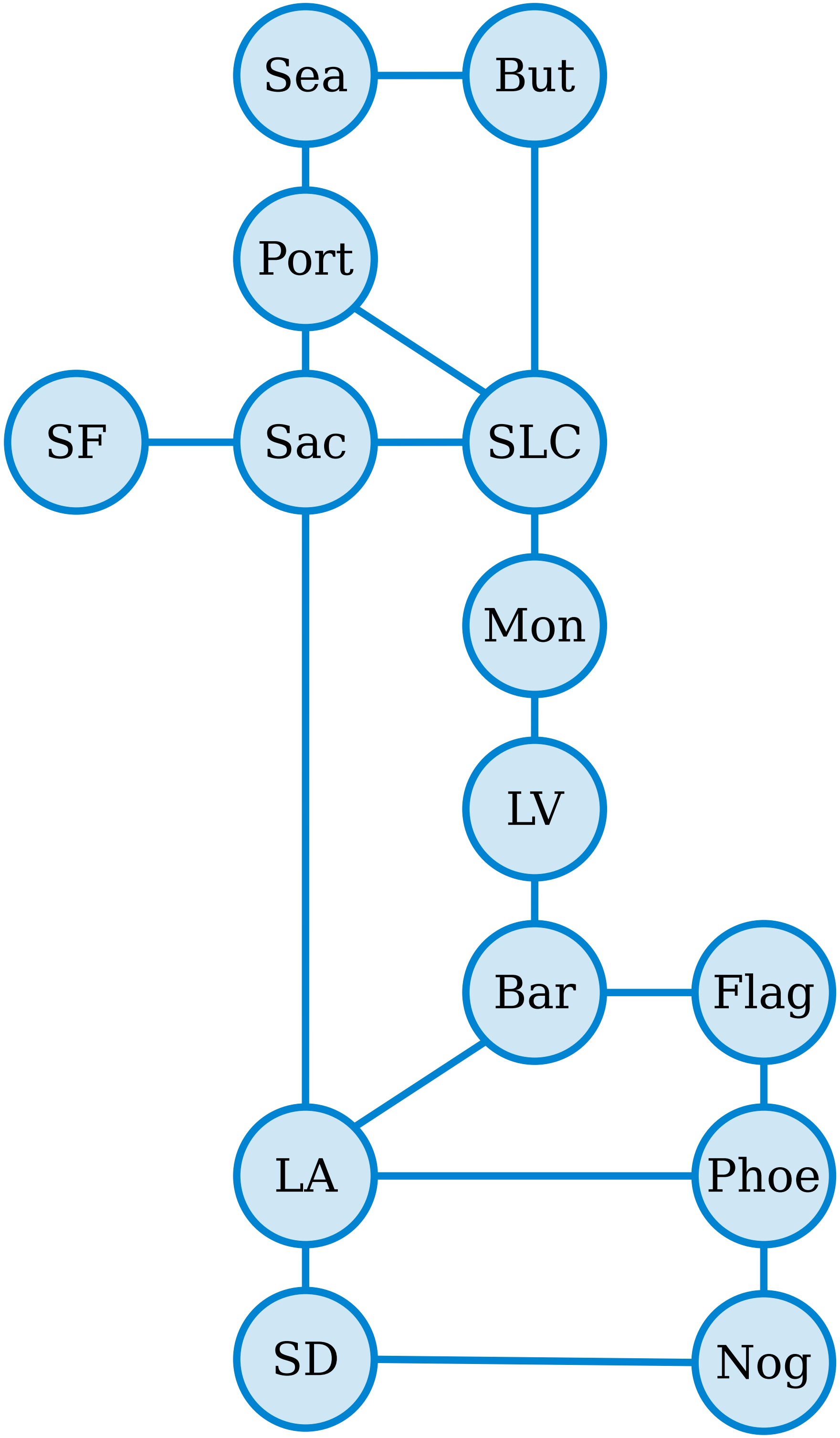

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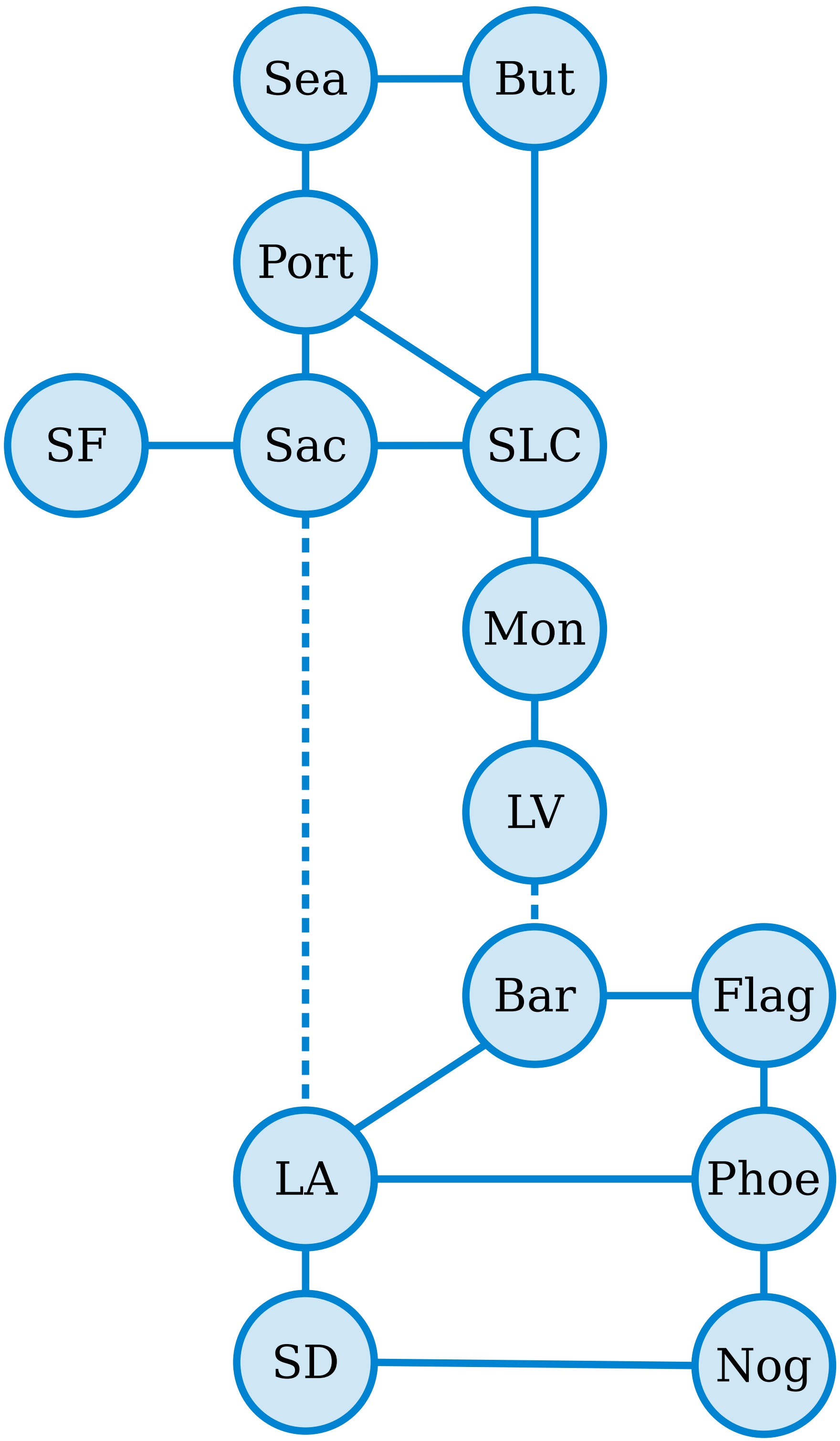

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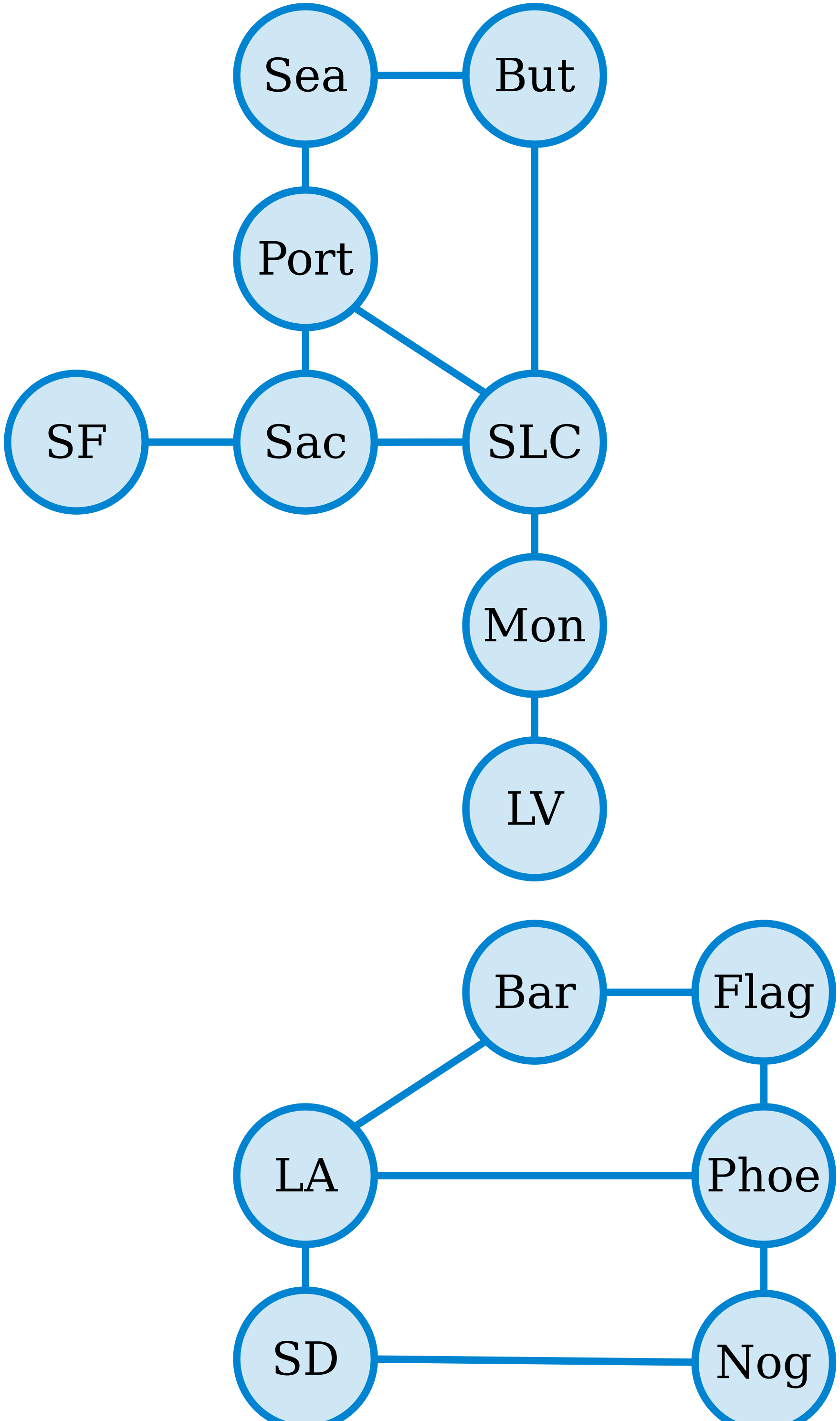

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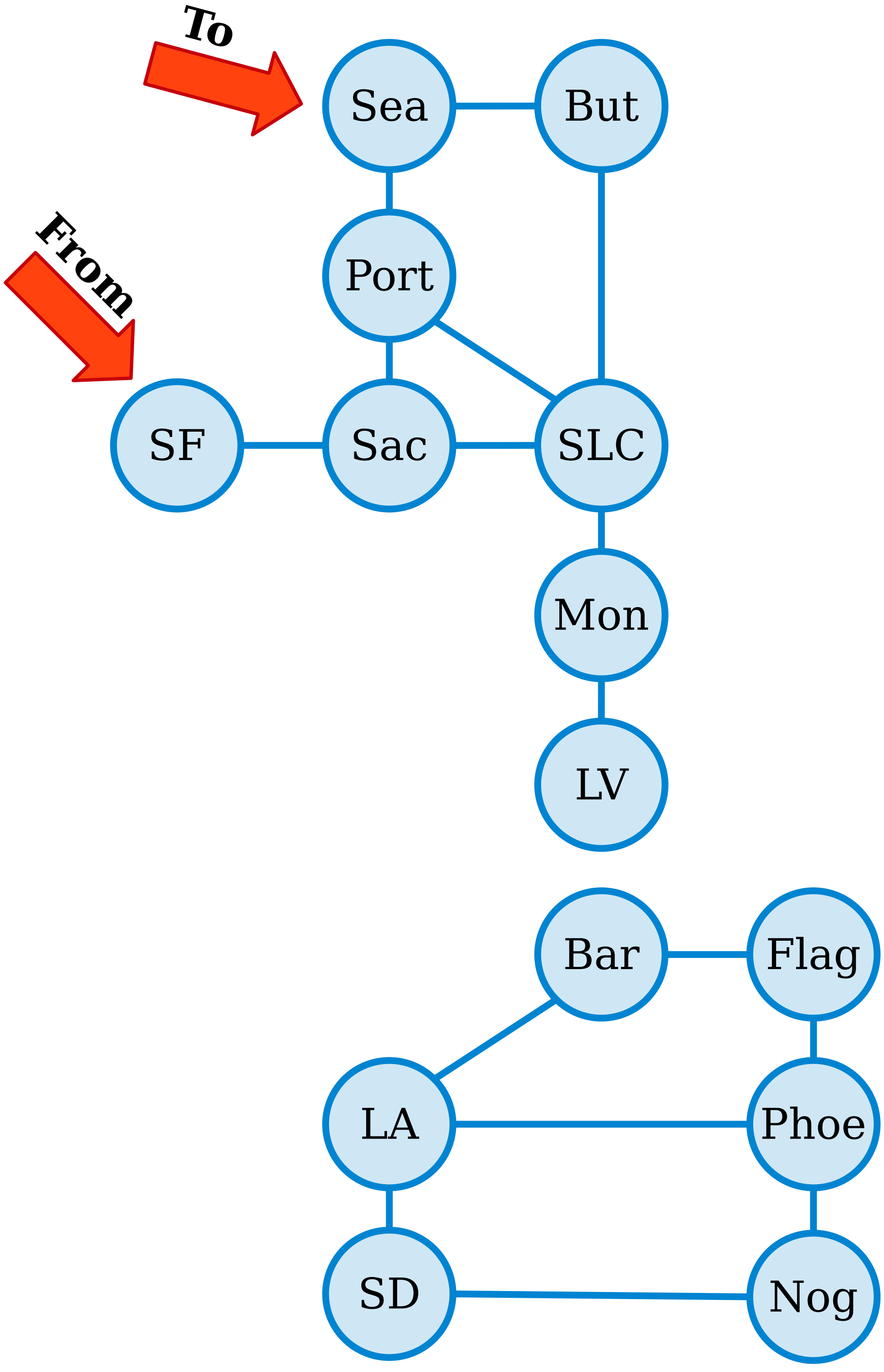

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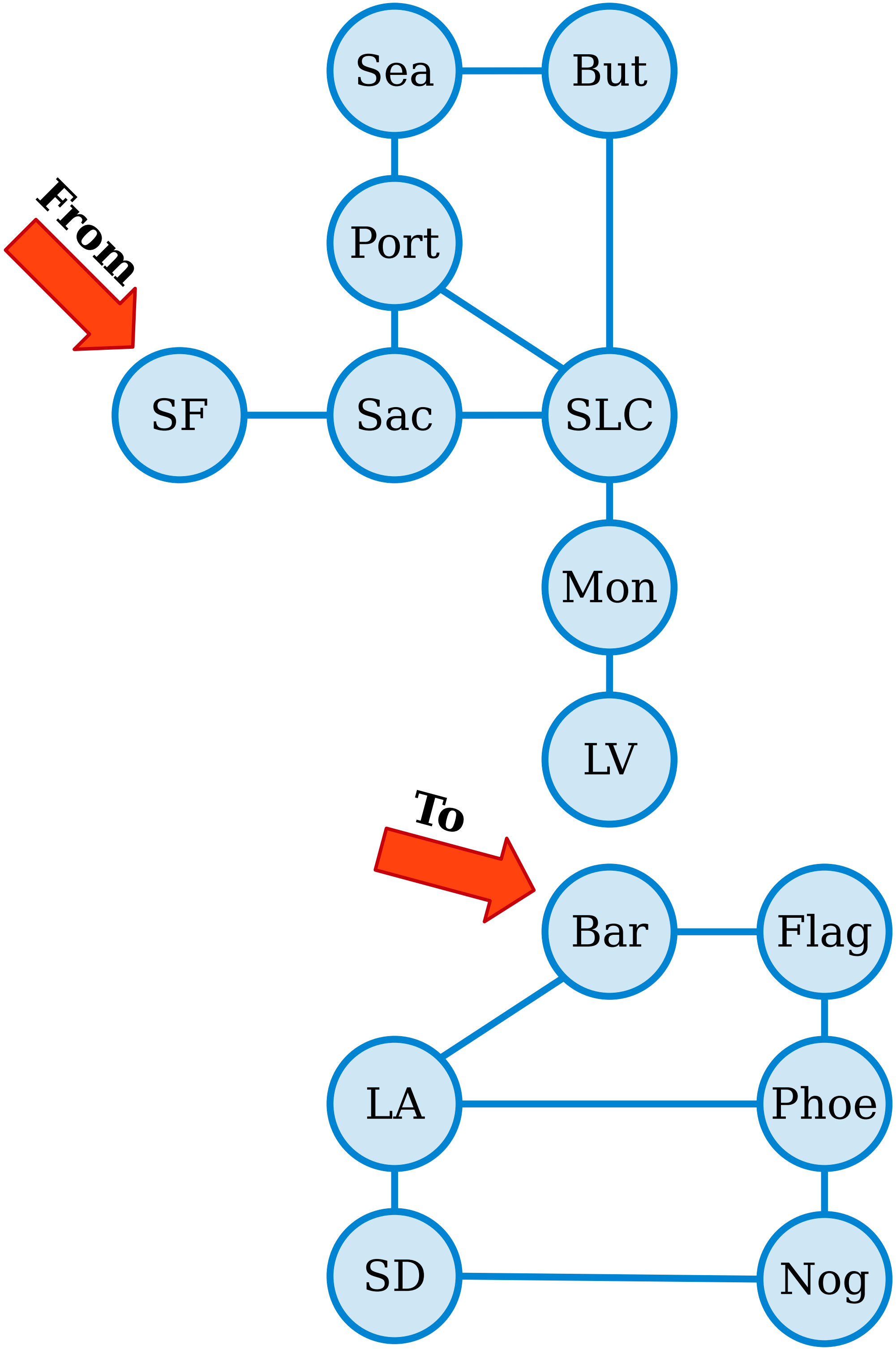

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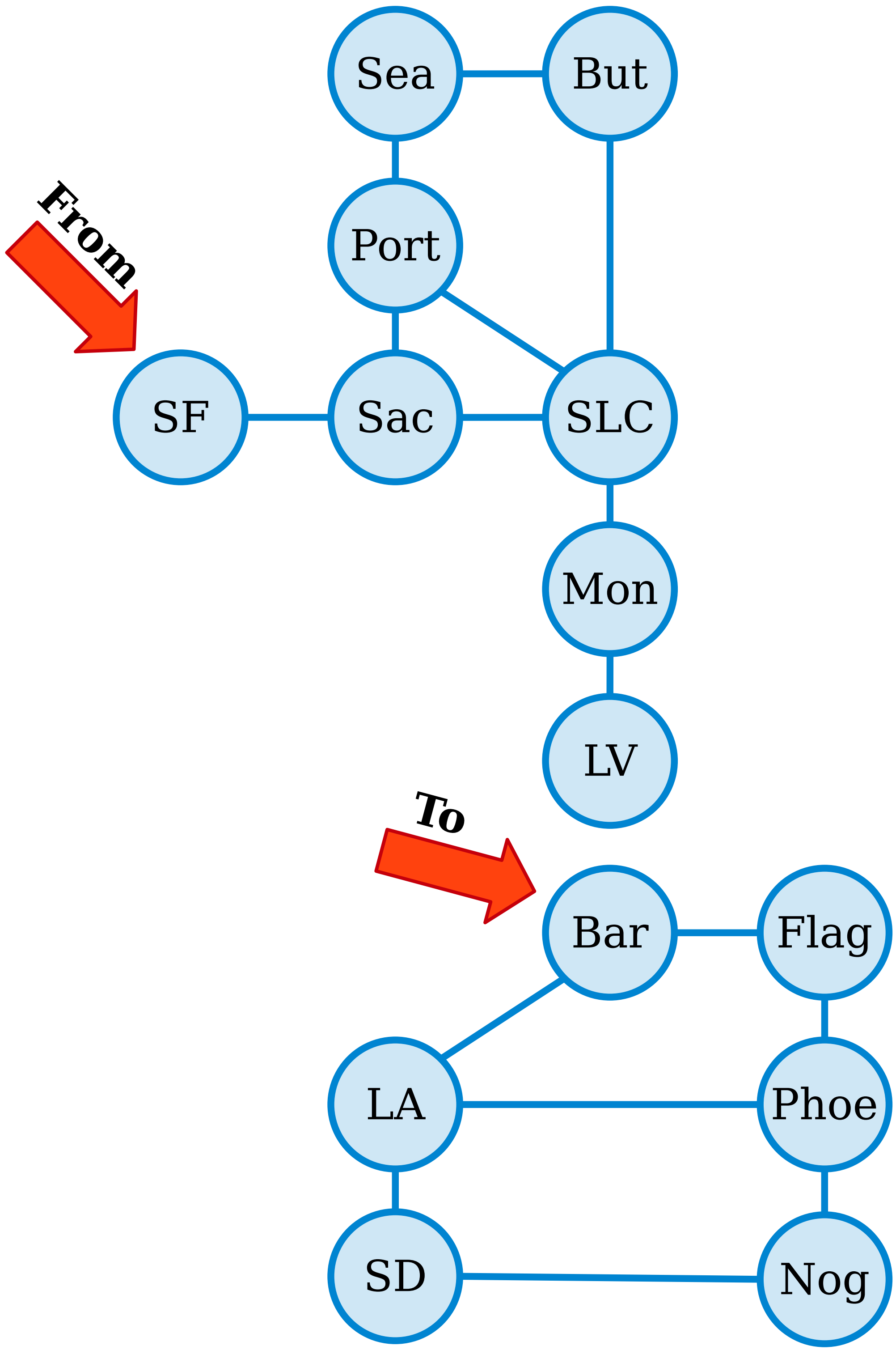

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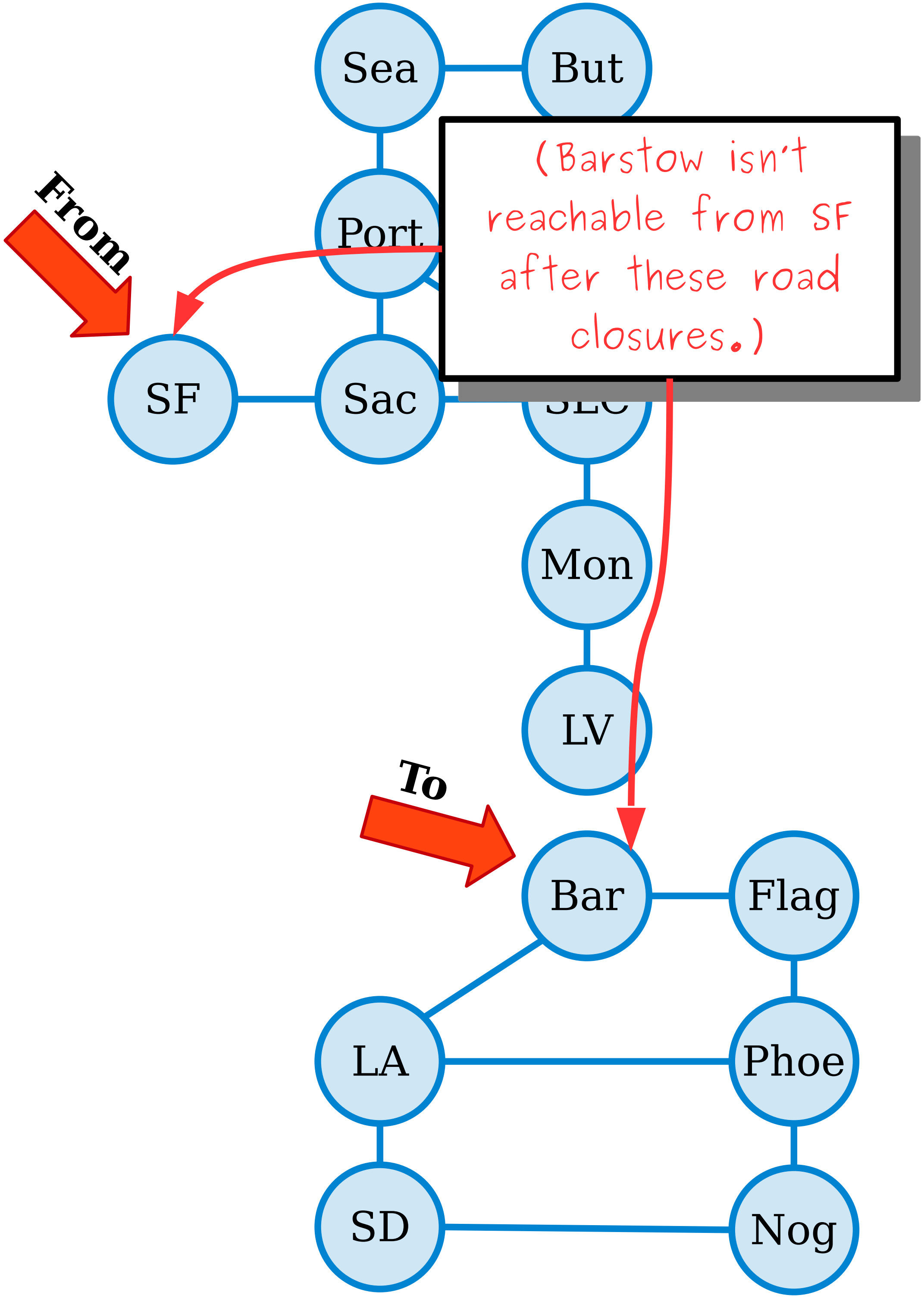

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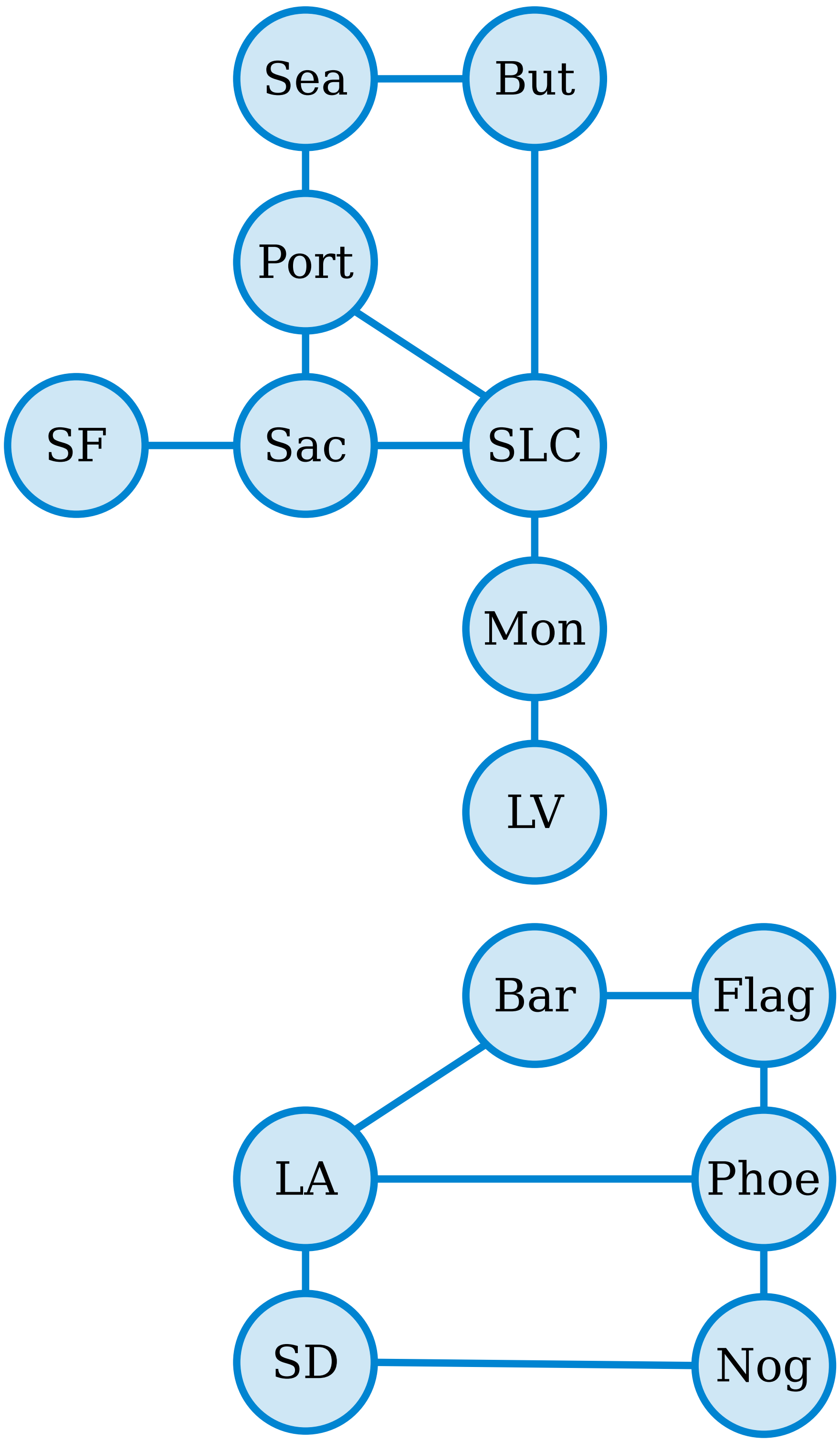

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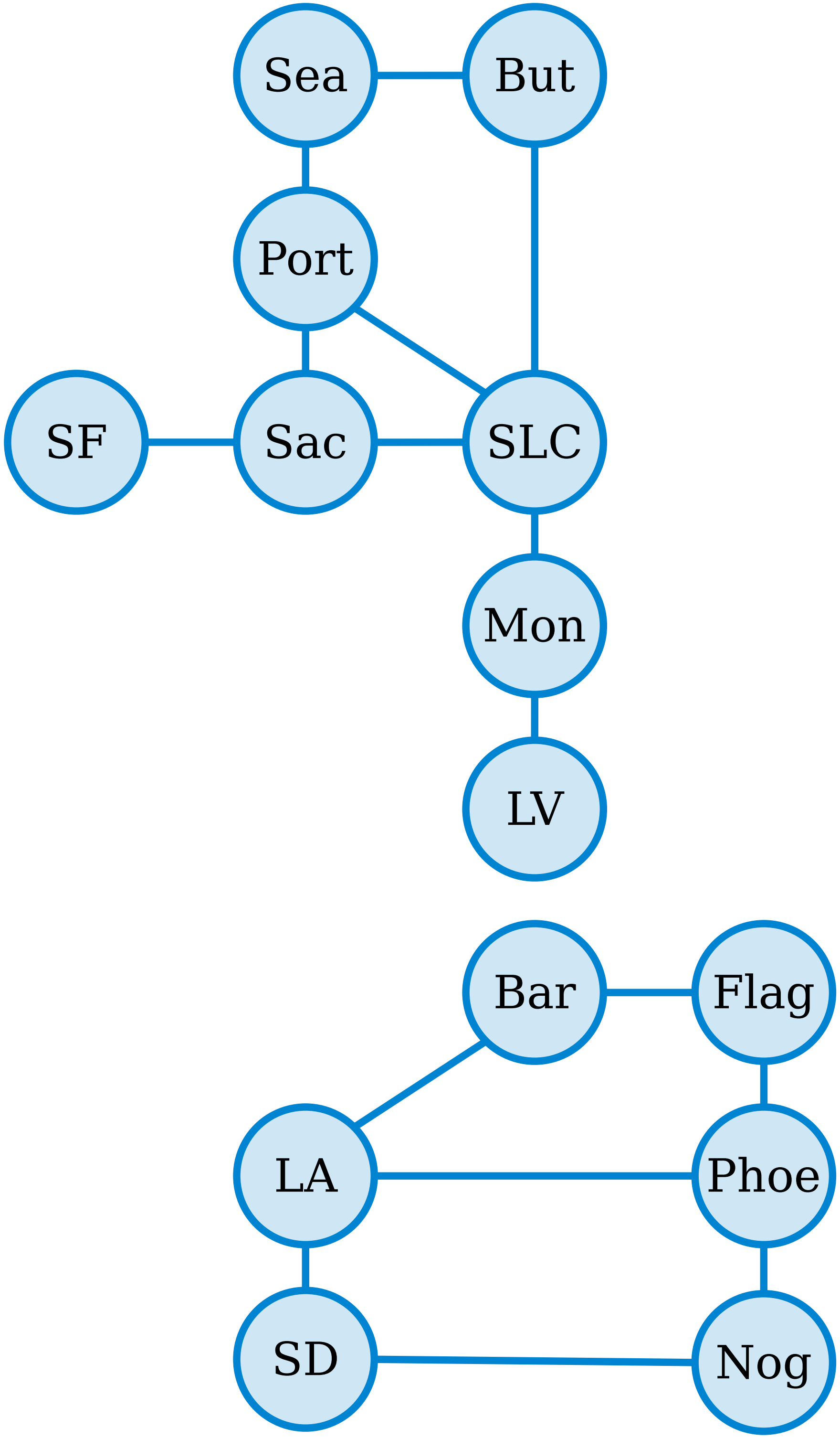

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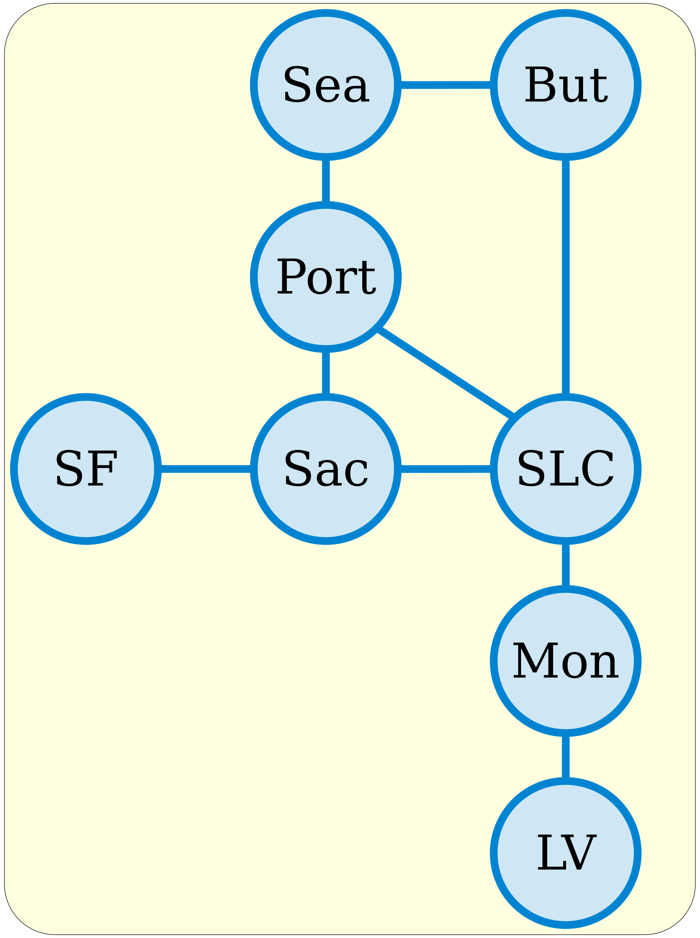
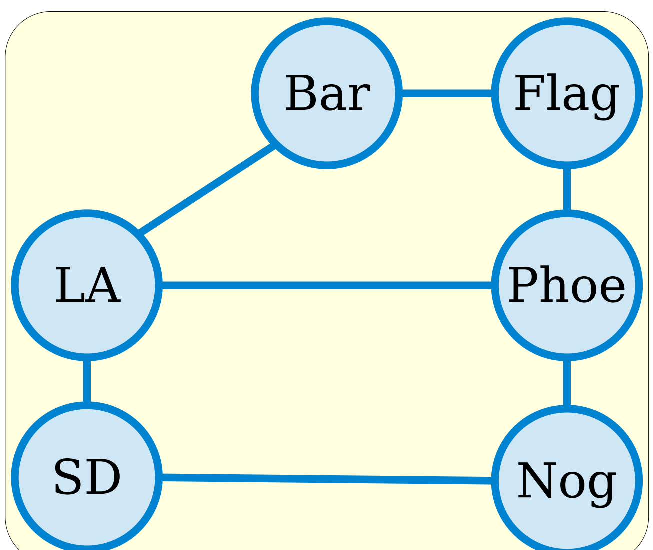
(This graph is not connected.)





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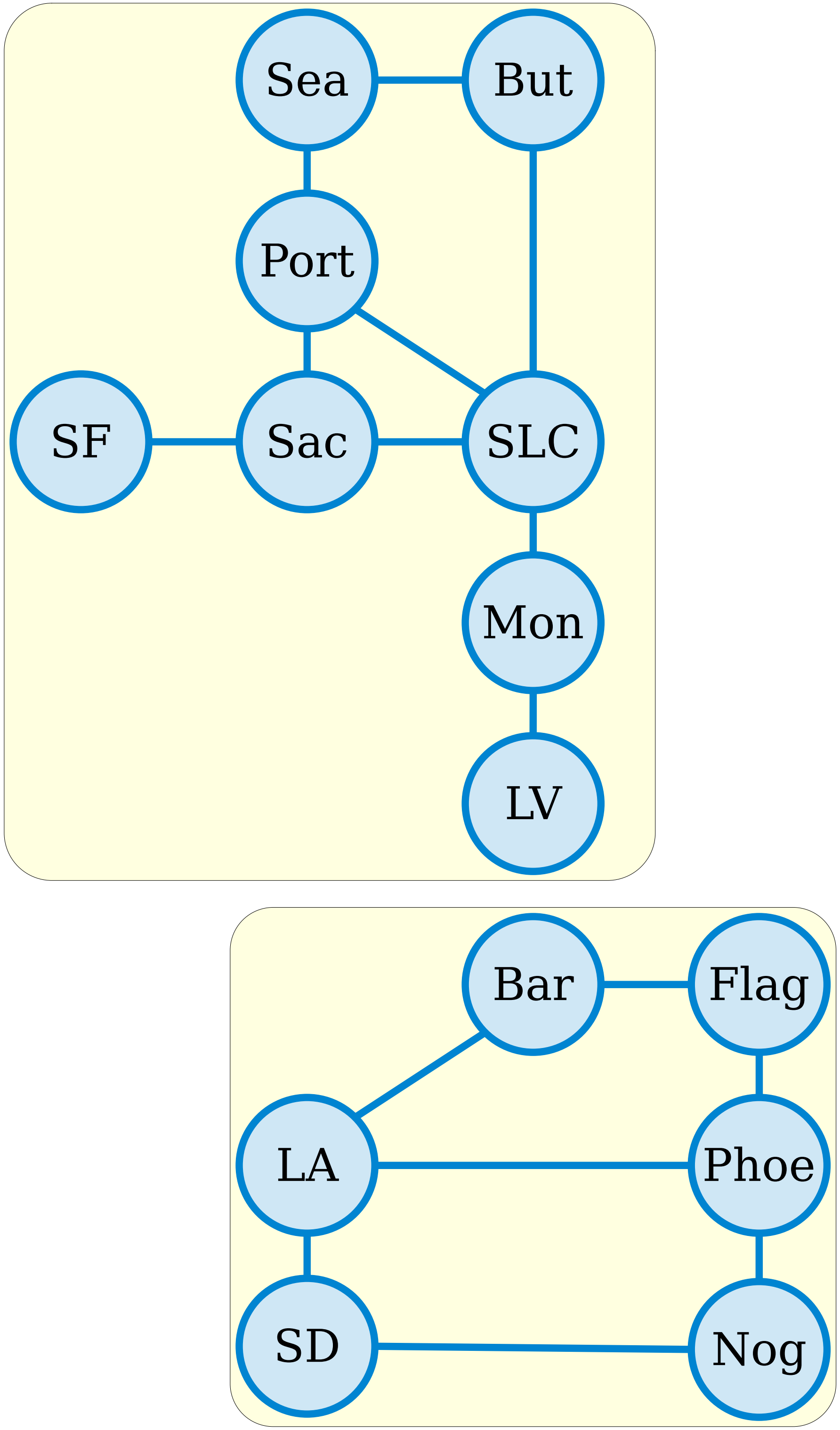

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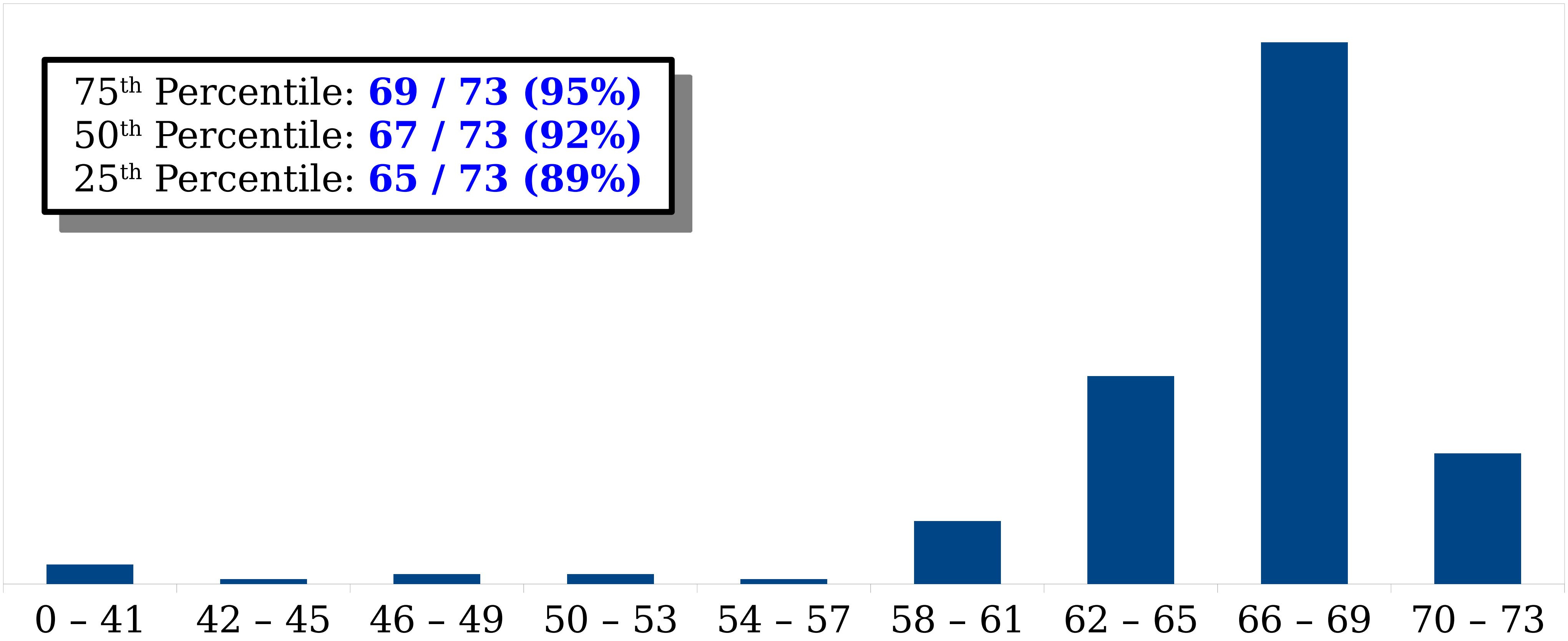
A ***connected component*** (or ***CC***) of G is a set consisting of a node and every node reachable from it.

Fun Facts

- Here's a collection of useful facts about graphs that you can take as a given.
  - ***Theorem:*** If $G = (V, E)$ is a (directed or undirected) graph and $u, v \in V$, then there is a path from u to v if and only if there's a walk from u to v.
  - ***Theorem:*** If G is an undirected graph and C is a cycle in G, then C's length is at least three and C contains at least three nodes.
  - ***Theorem:*** If $G = (V, E)$ is an undirected graph, then every node in V belongs to exactly one connected component of G.
  - ***Theorem:*** If $G = (V, E)$ is a (directed or undirected) graph and $u, y_0, y_1, \ldots, y_m, v$ is a walk from u to v and $v, z_0, z_1, \ldots, z_n, x$ is a walk from v to x, then $u, y_0, y_1, \ldots, y_m, v, z_0, z_1, \ldots, z_n, x$ is a walk from u to x.
- Looking for more practice working with formal definitions? Prove these results!

Time-Out for Announcements!

Problem Set Two Graded



Midterm Exam Logistics

- Our first midterm exam is next ***Monday, October 20th*** from ***7:00PM – 10:00PM***.
  - ***Seating assignments are available***. Write your seat number down in case the WiFi cuts out before the exam.
- You're responsible for Lectures 00 – 05 and topics covered in PS1 – PS2.
  - Later lectures (functions forward) and problem sets (PS3 onward) won't be tested here.
  - Exam problems may build on the written or coding components from the problem sets.
- The exam is closed-book, closed-computer, and limited-note. You can bring a double-sided, 8.5" × 11" sheet of notes with you to the exam, decorated however you'd like.

Preparing for the Exam

- Your amazing CA Ari is holding a review session this Friday from 3PM – 4PM in CoDa E160.
- Make sure to ***review your feedback*** on PS1 and PS2.
  - "Make new mistakes."
  - Come talk to us if you have questions!
- There's a huge bank of practice problems up on the course website.
- Best of luck – ***you can do this!***

Participation Opt-Out

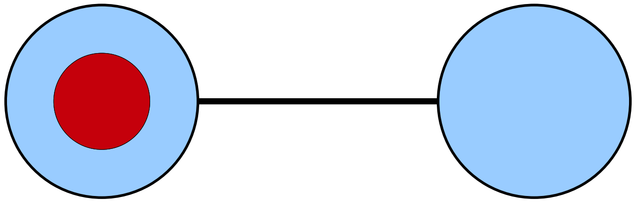
- By default, all on-campus students have 5% of their grade allocated from lecture attendance and participation.
- If you are an on-campus student and want to opt out, shifting that 5% onto your final exam, ***fill out the opt-out form on Ed by Friday at 11:59 PM***.

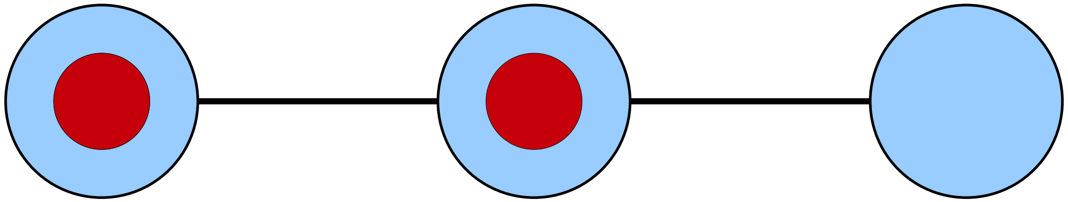
Back to CS103!

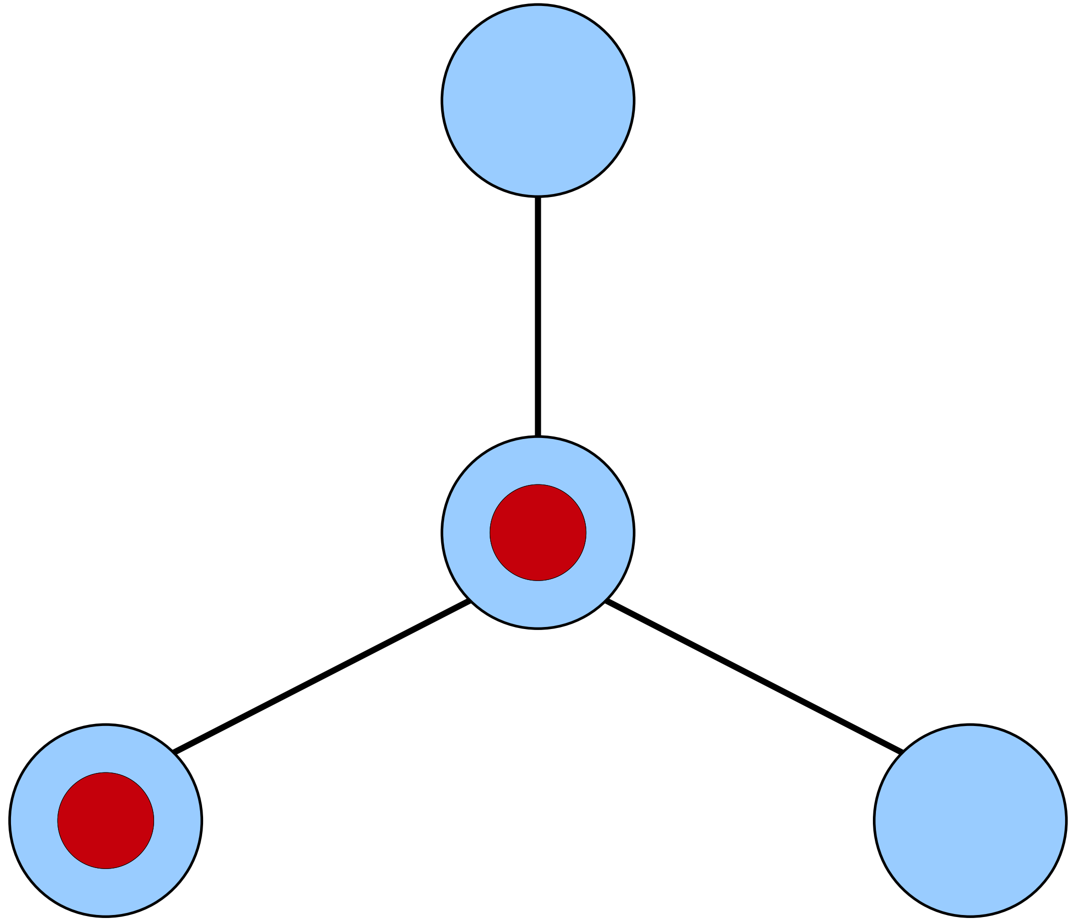
Application: ***Local Area Networks***

The Internet and LANs

- The internet consists of several separate ***local area networks*** (***LANs***) that are "internetworked" together.
- Local area networks cover small areas – a single hallway in a dorm, an office building, a college campus, etc.
- The internet then links those smaller LANs into one giant network where everyone can talk to everyone.
- ***Focus for today:*** How do messages flow through a LAN?

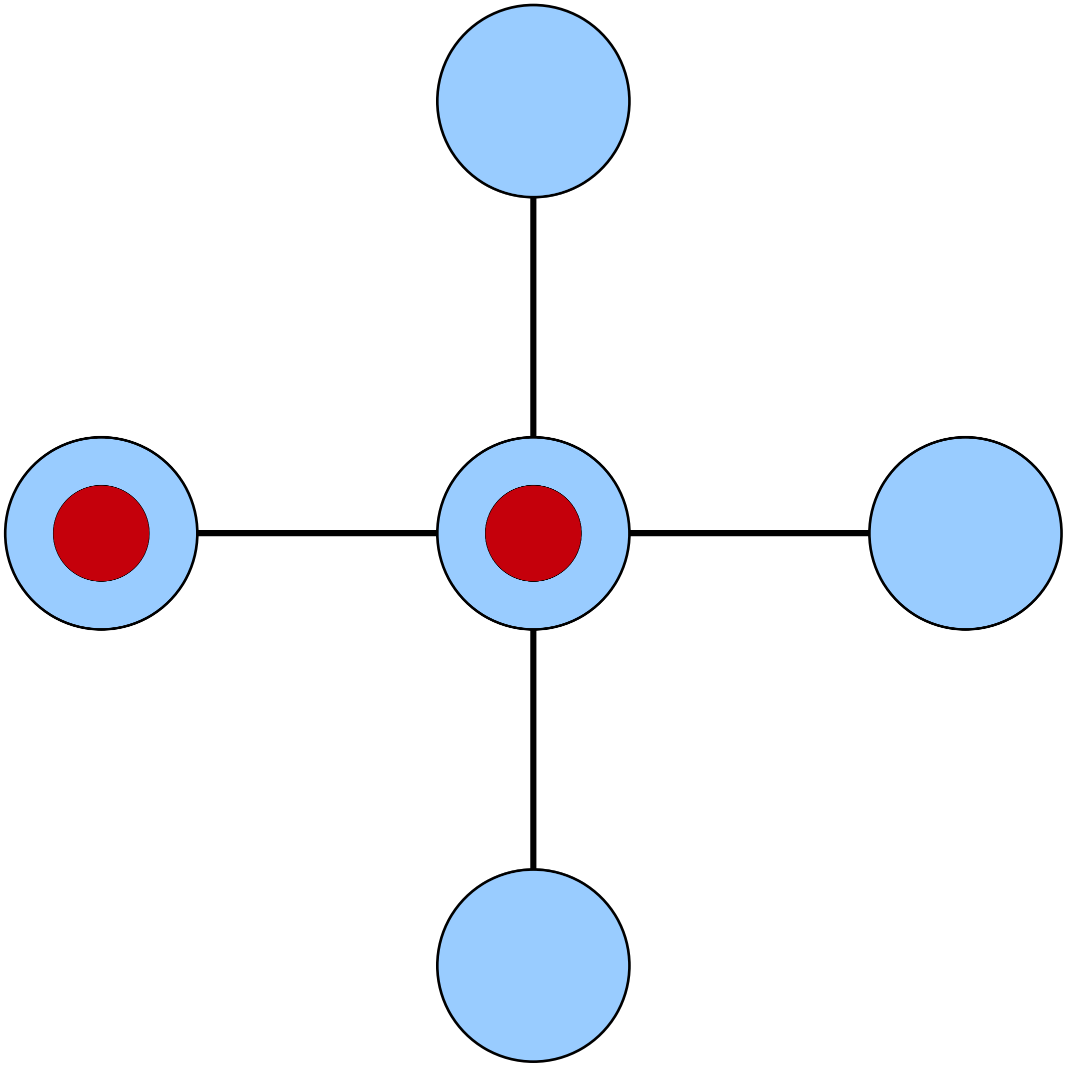


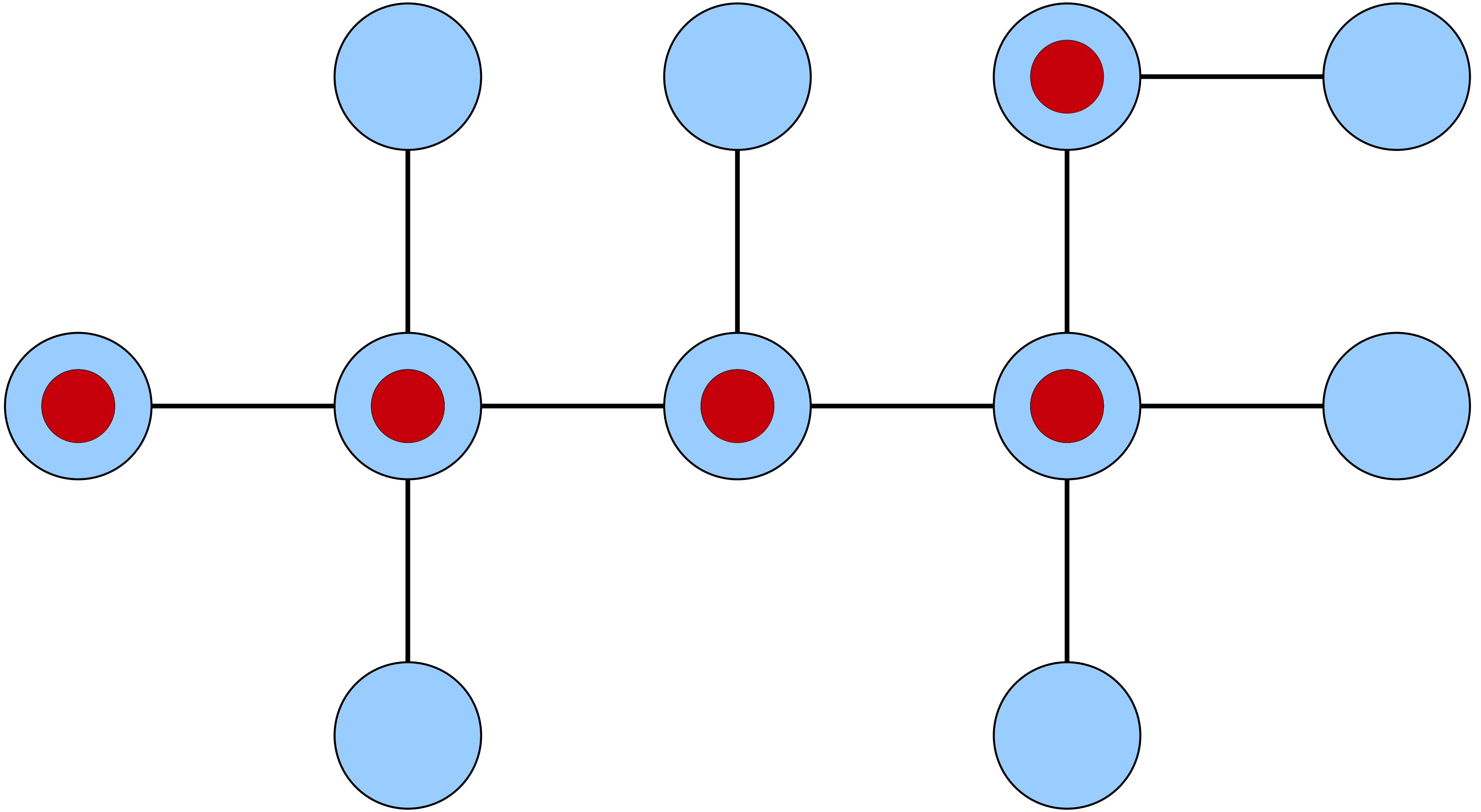




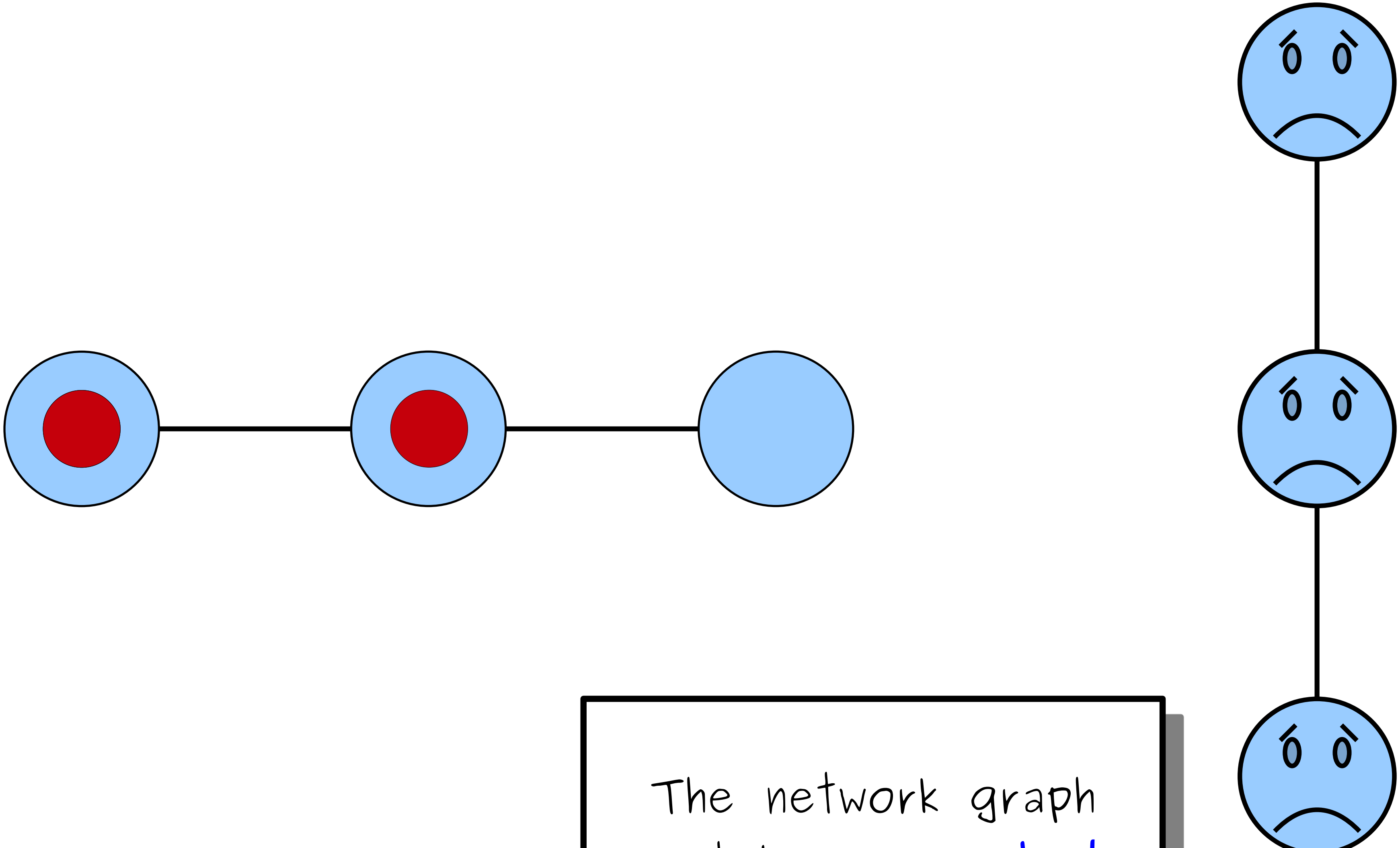
Message Movement

- When a computer receives a message, it repeats that message on all its links except the one it received the message on.
- The computers don’t inspect the message contents or try to be clever – it’s purely “came in on link *X*, goes out on all links but *X*.”

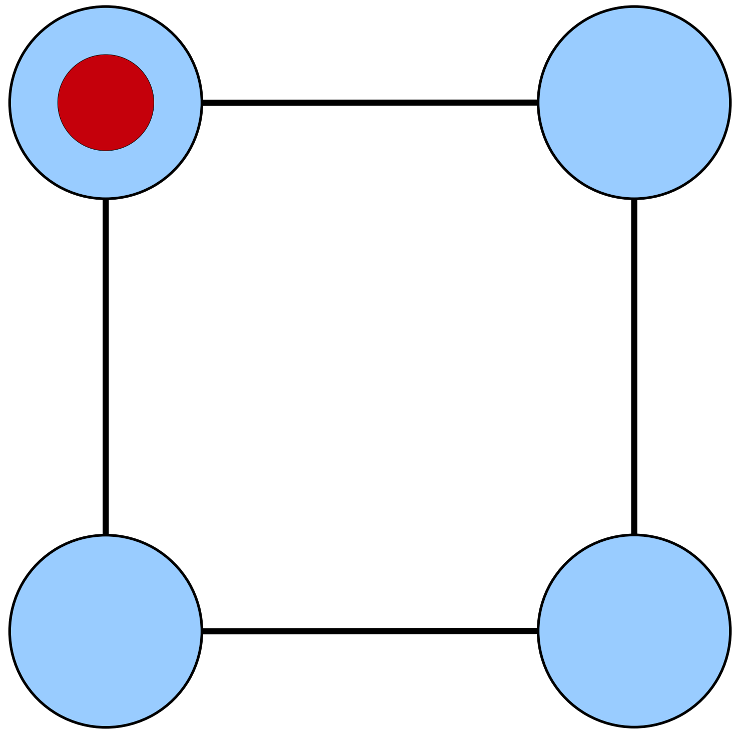




Two Pitfalls

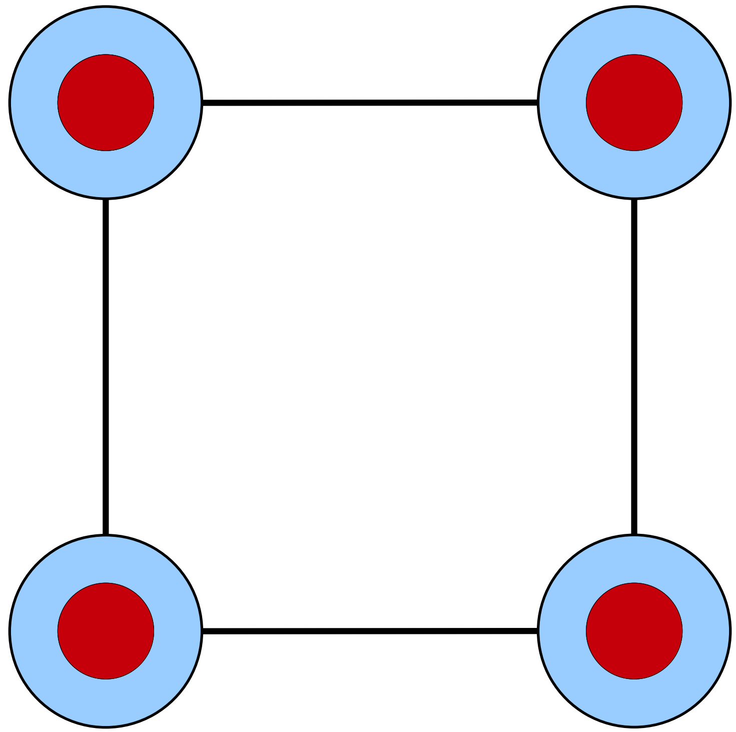


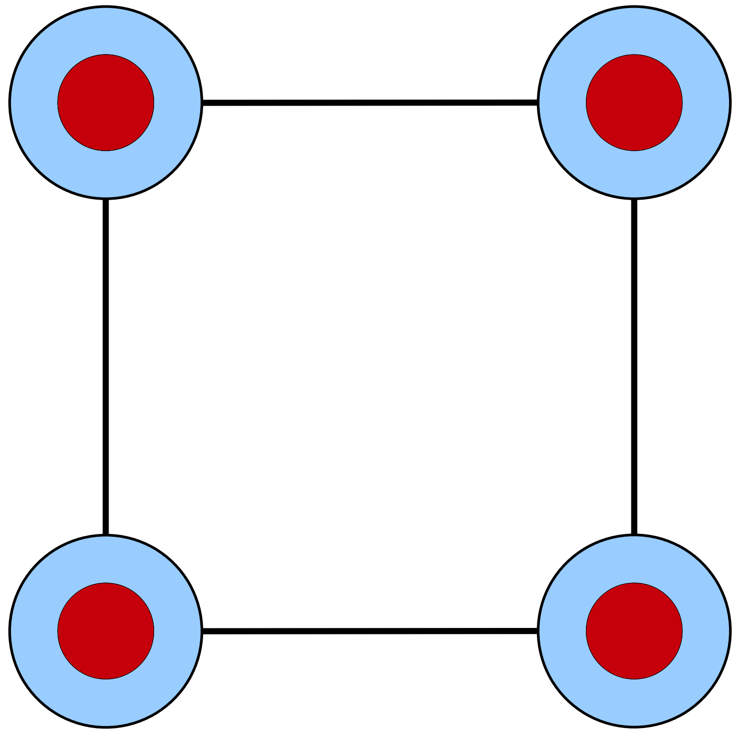
The network graph must be connected.

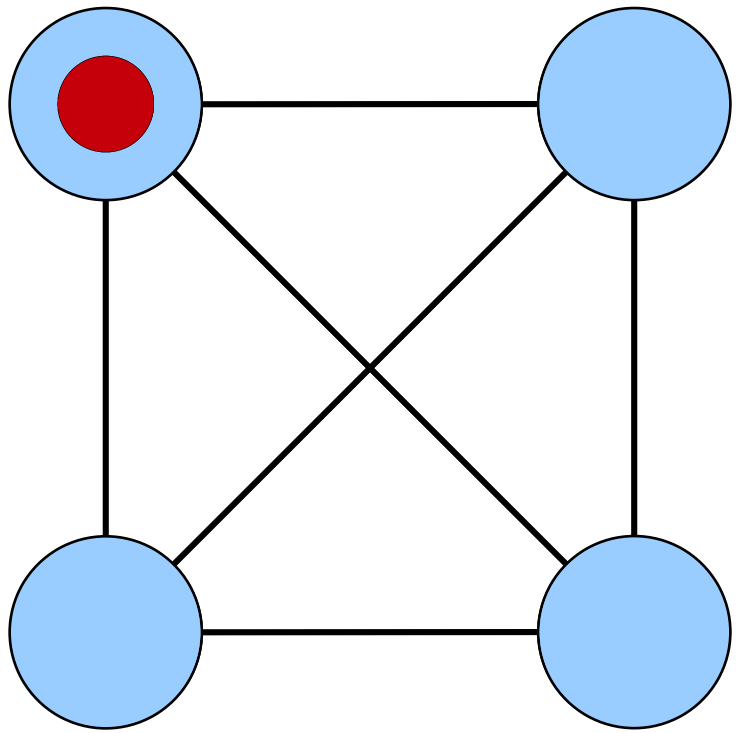


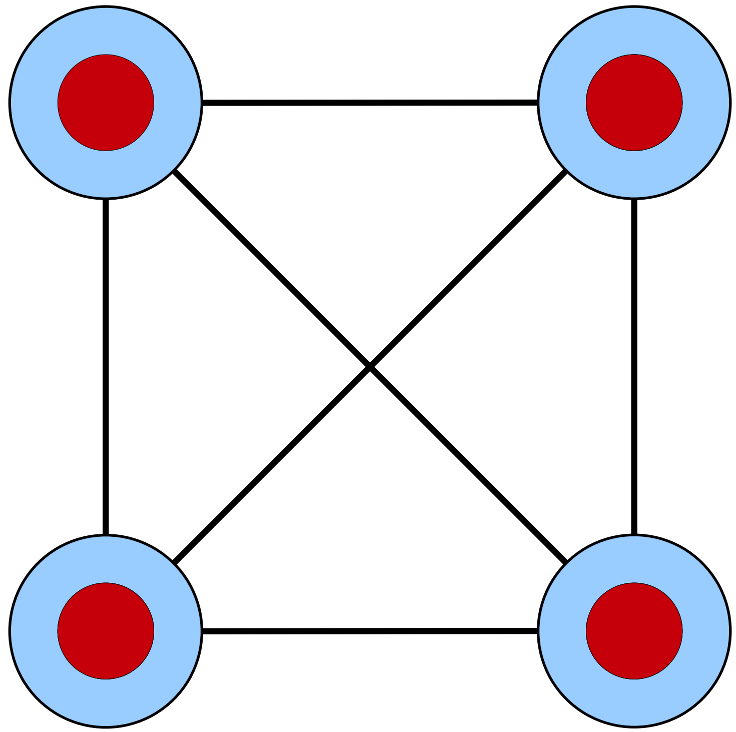
What will happen if this computer sends a message through the network?

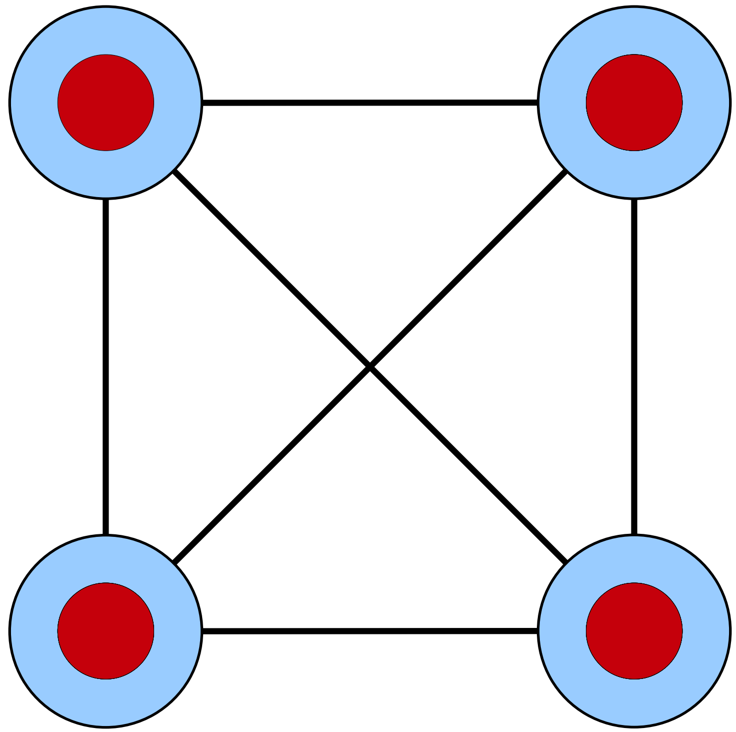
Answer at **https://cs103.stanford.edu/pollev**







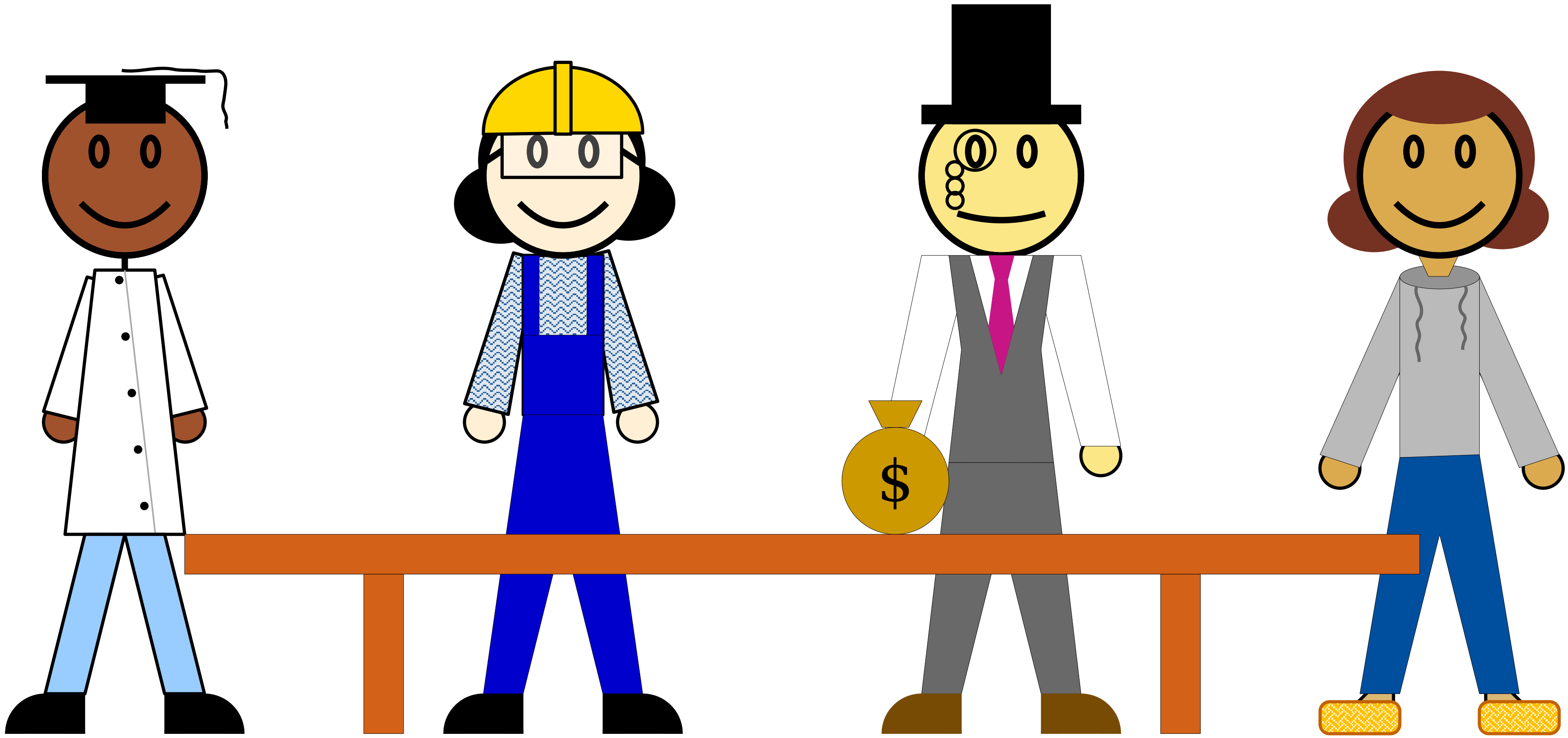




Broadcast Storms

- A ***broadcast storm*** occurs when there's a cycle in the network graph.
- A single message can repeat forever, or exponentially amplify until the network fails.
- ***Solution:*** Don't let the network graph have any cycles.
- A graph $G = (V, E)$ is called ***acyclic*** if it has no cycles.

You have a collection of computers that need to be wired up into a LAN. How should you choose the shape of the network?



CTO

Connected,
No Cycles

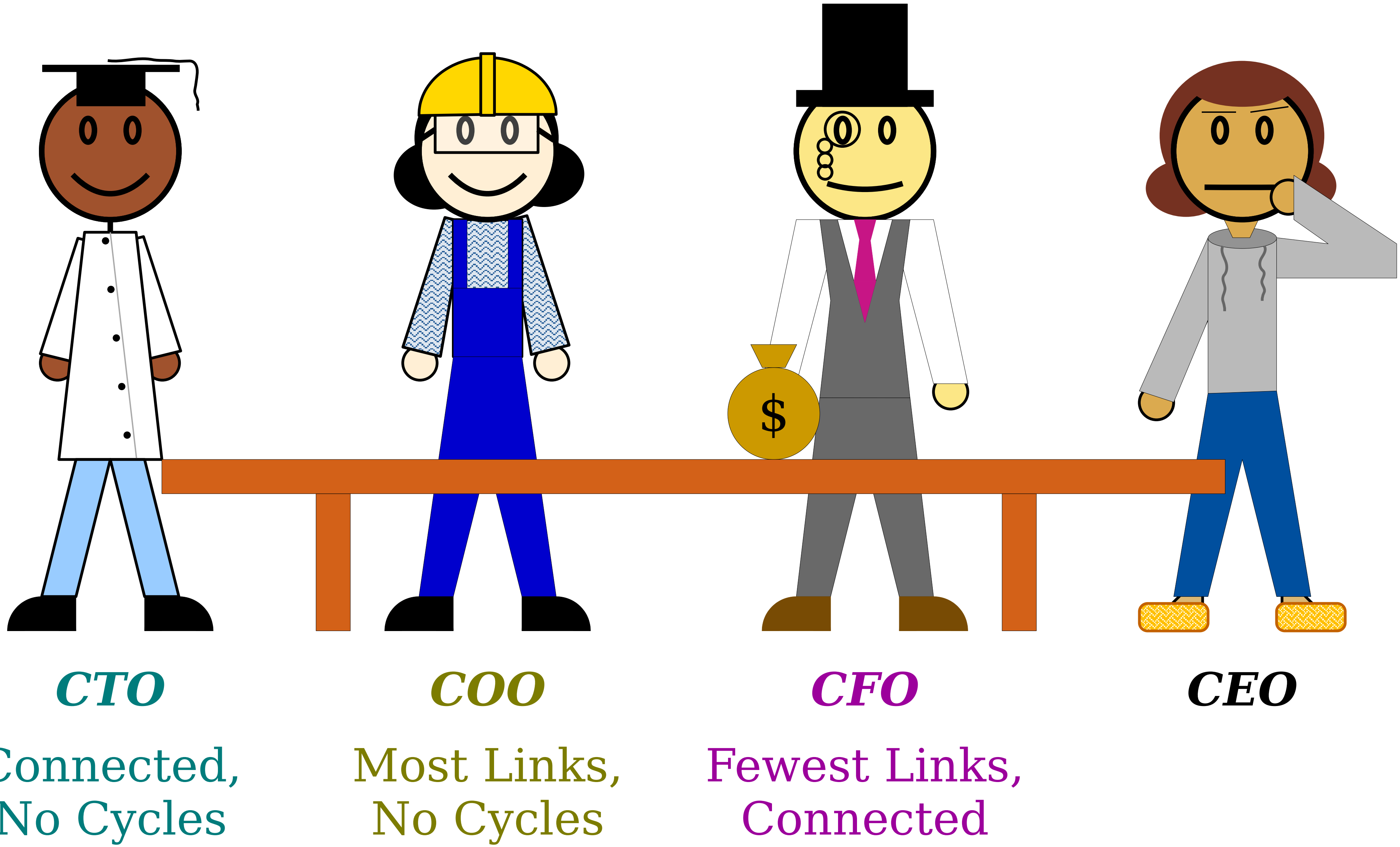
COO

Most Links,
No Cycles

CFO

Fewest Links,
Connected

CEO


CTO
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CFO
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CEO
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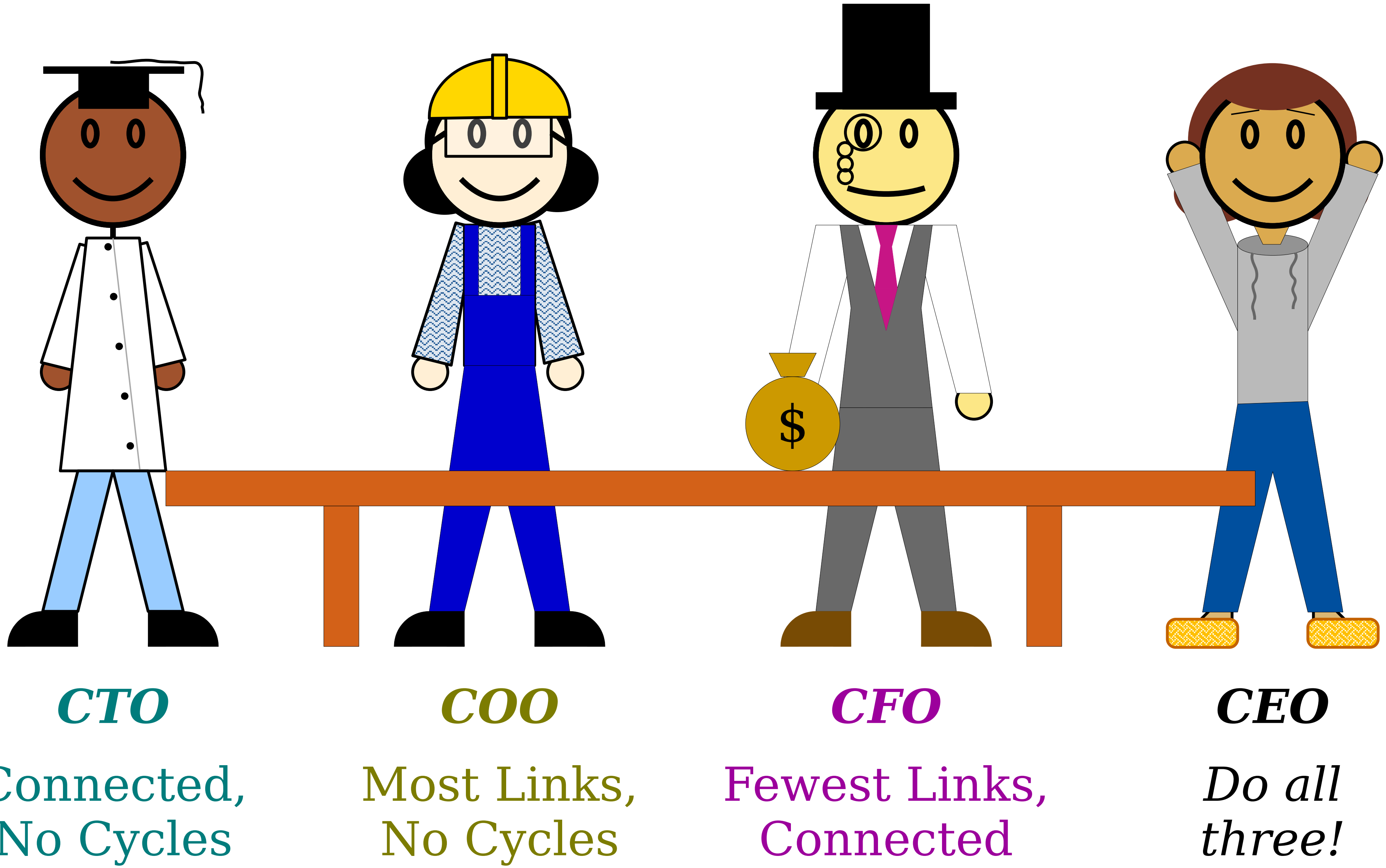
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CTO
Connected, No Cycles
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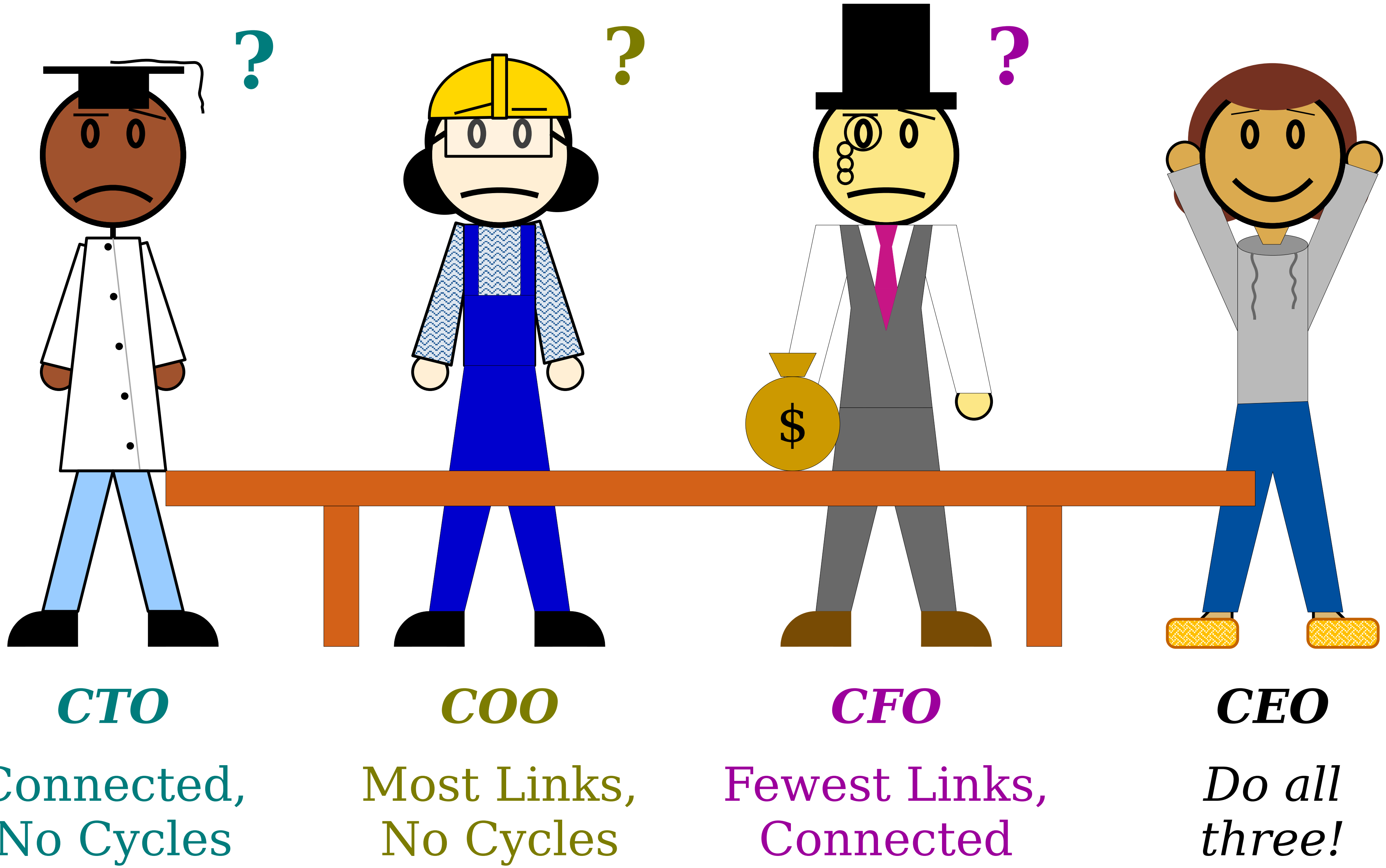
Fewest Links, Connected
CEO
Do all three!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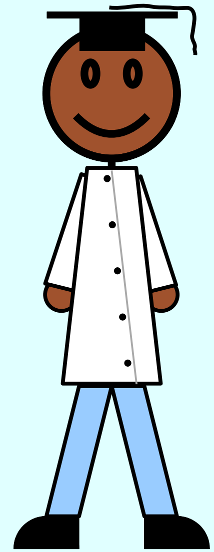
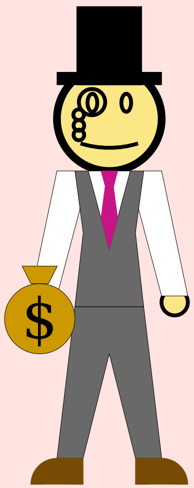


Connected, Acyclic

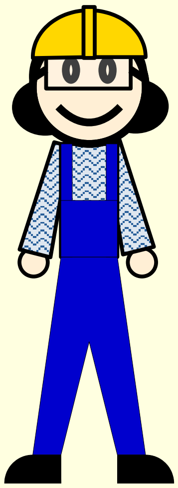


Minimally Connected

(Connected, but deleting any edge disconnects its endpoints.)

If *any* of these conditions hold, then *all* of these conditions hold.

A graph with any of these properties is called a ***tree***.



Maximally Acyclic

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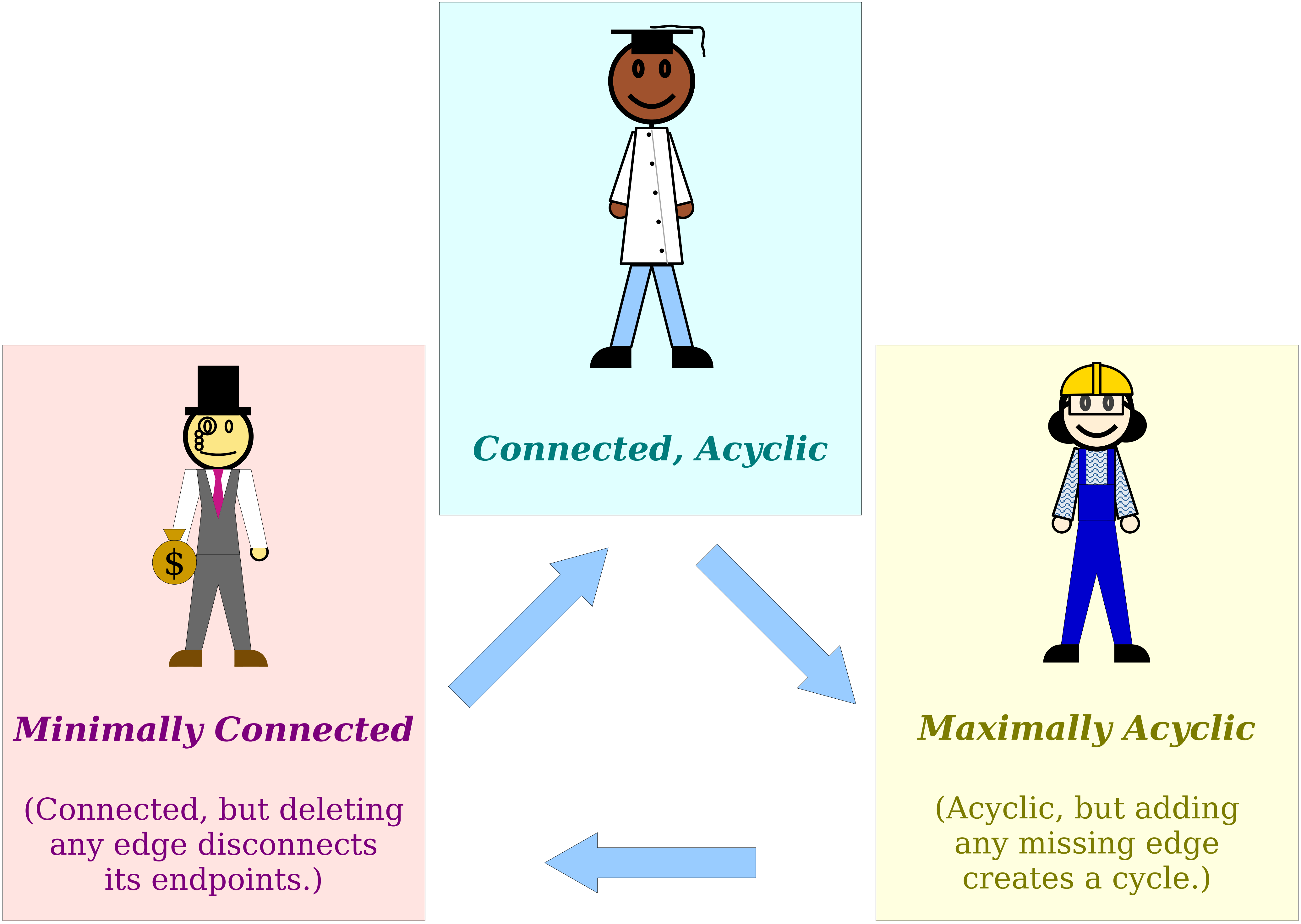
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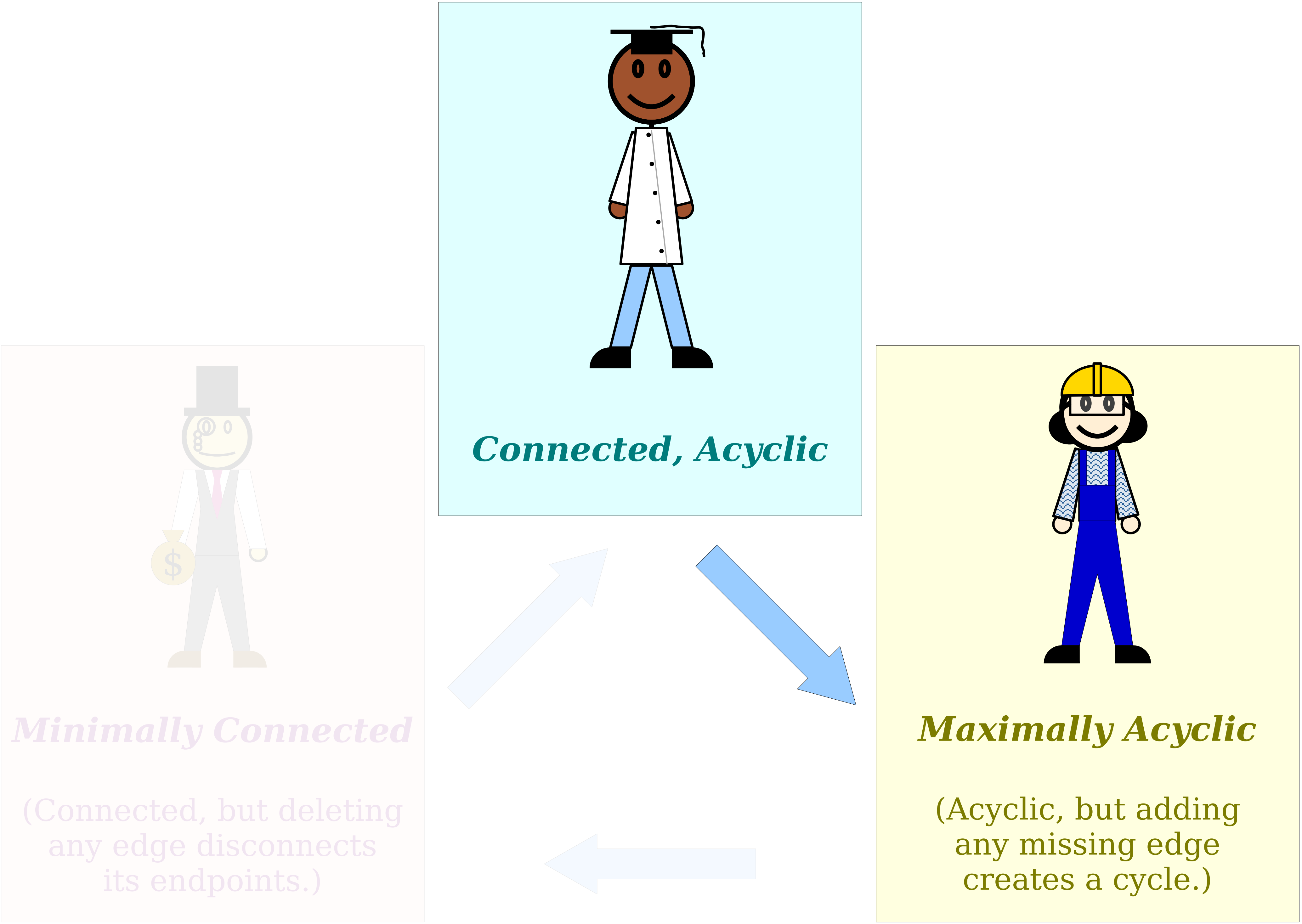
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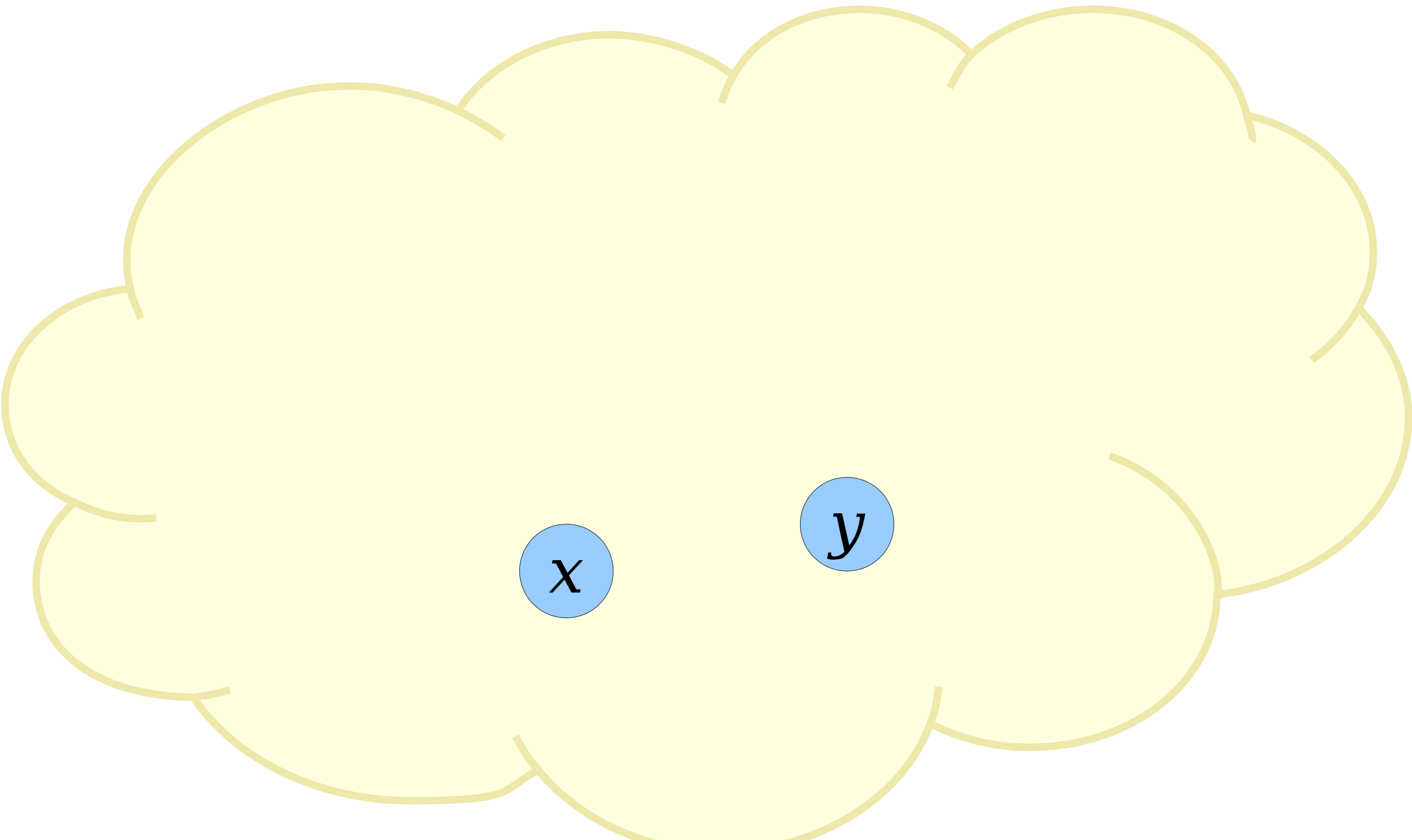

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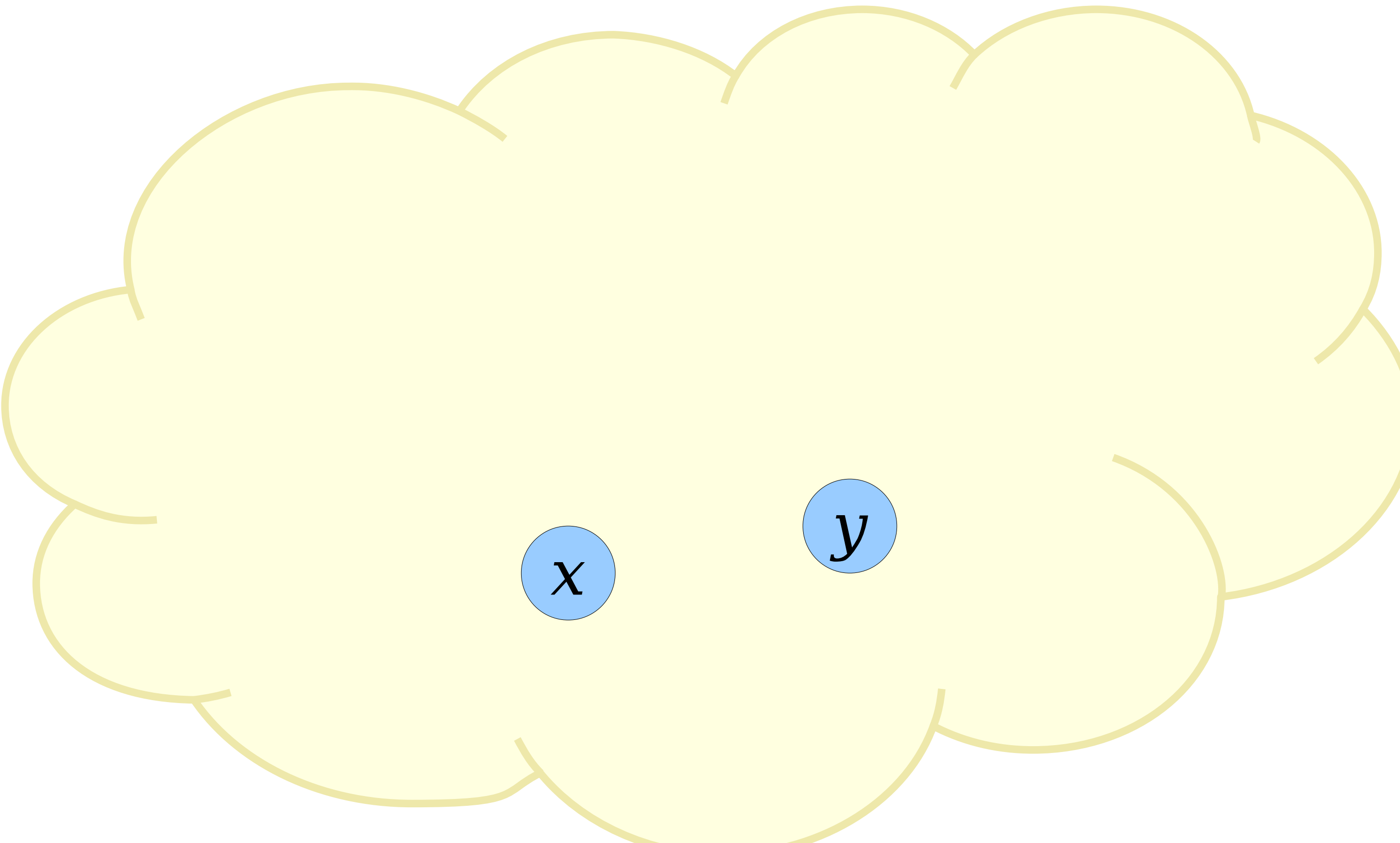

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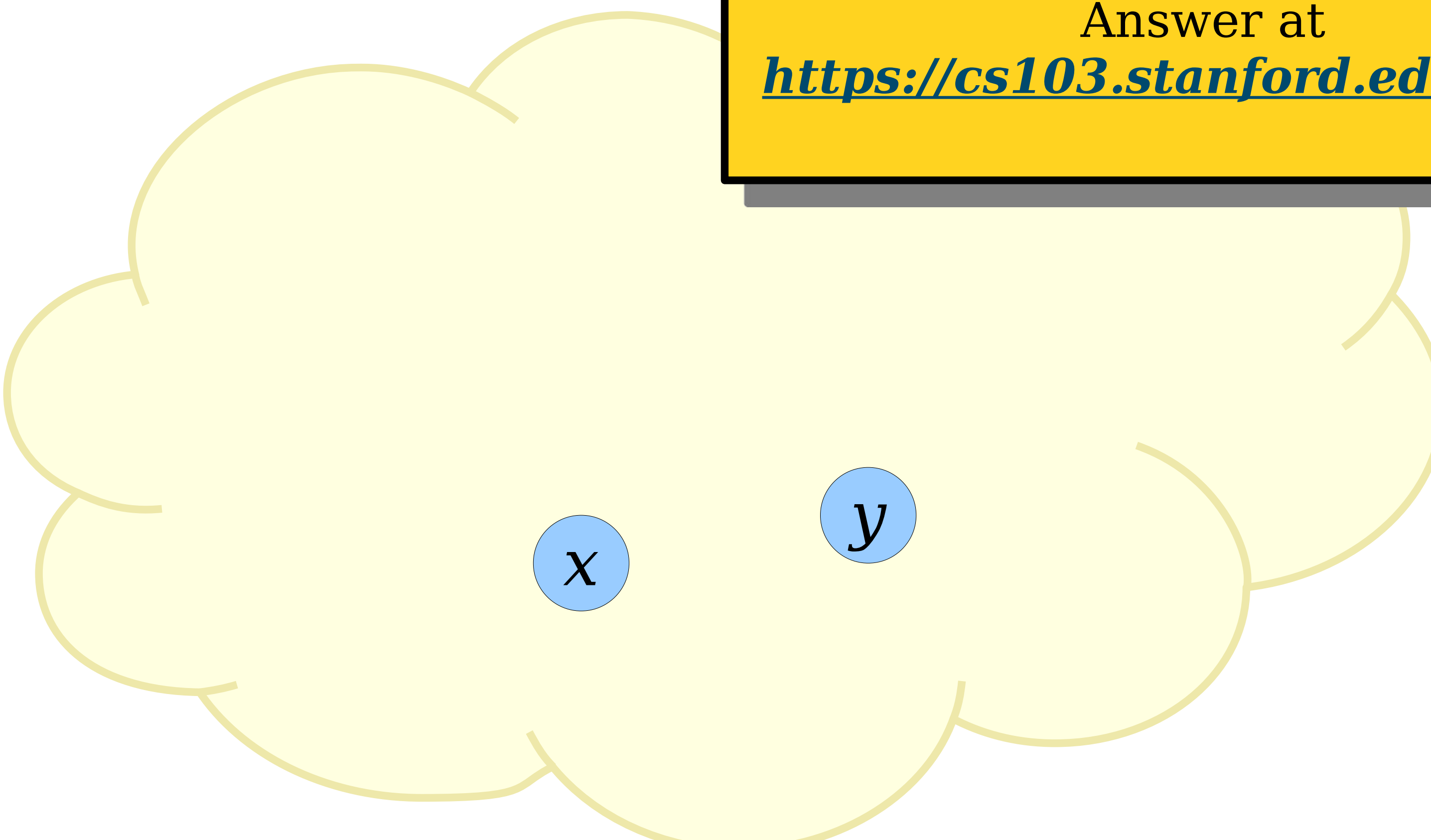
What do we know about x and y given that T is connected?

Answer at
https://cs103.stanford.edu/pollev



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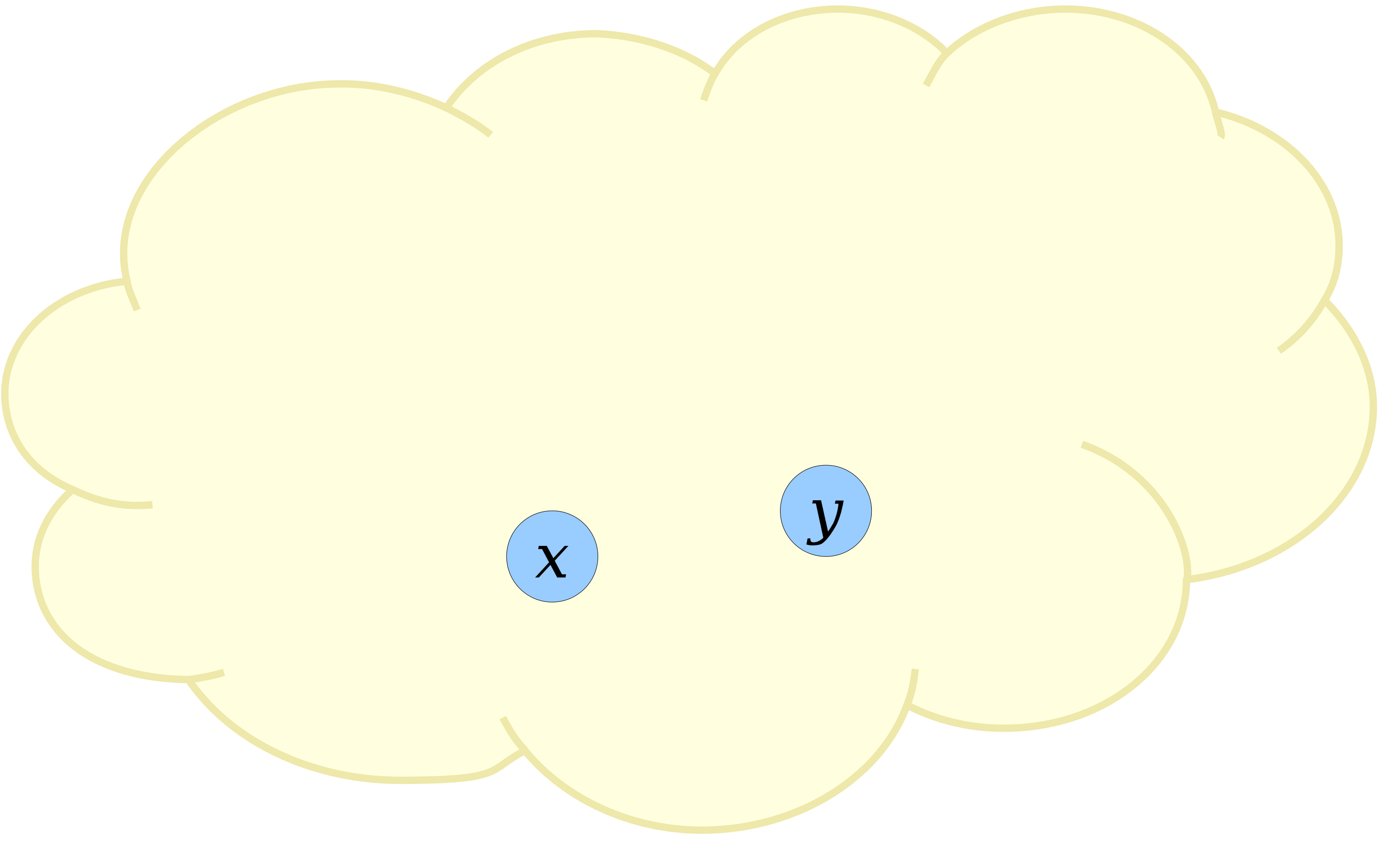

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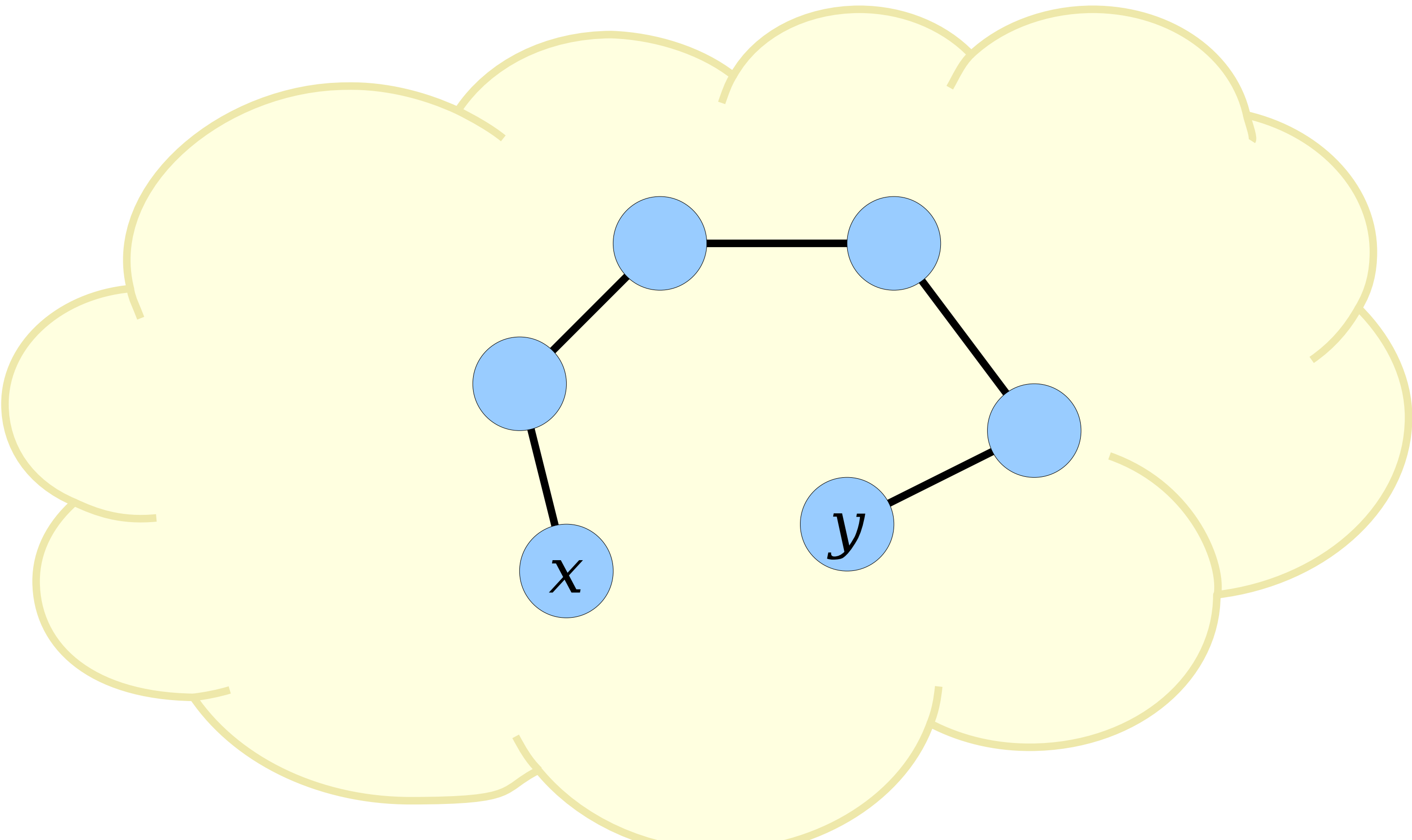

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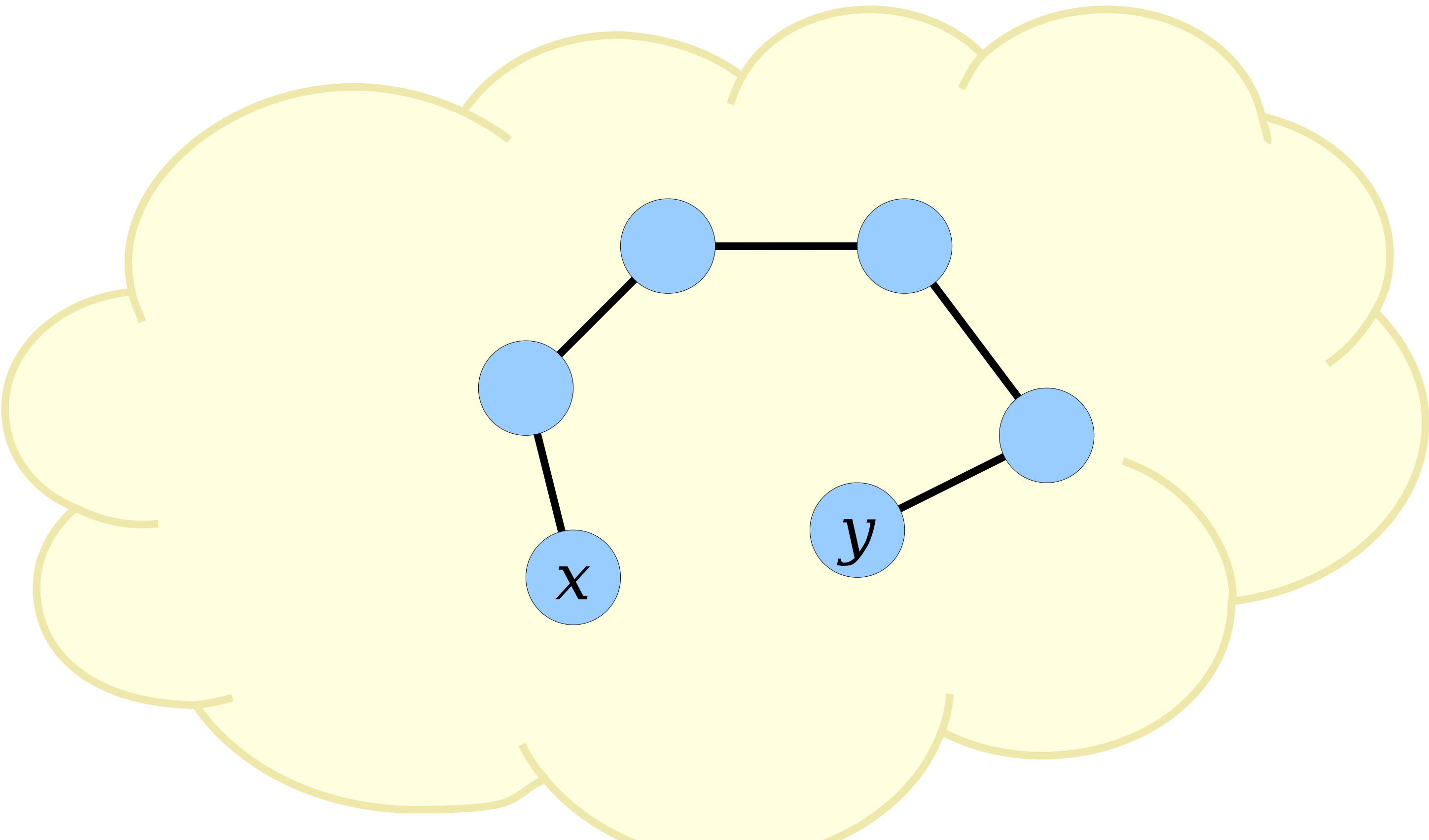

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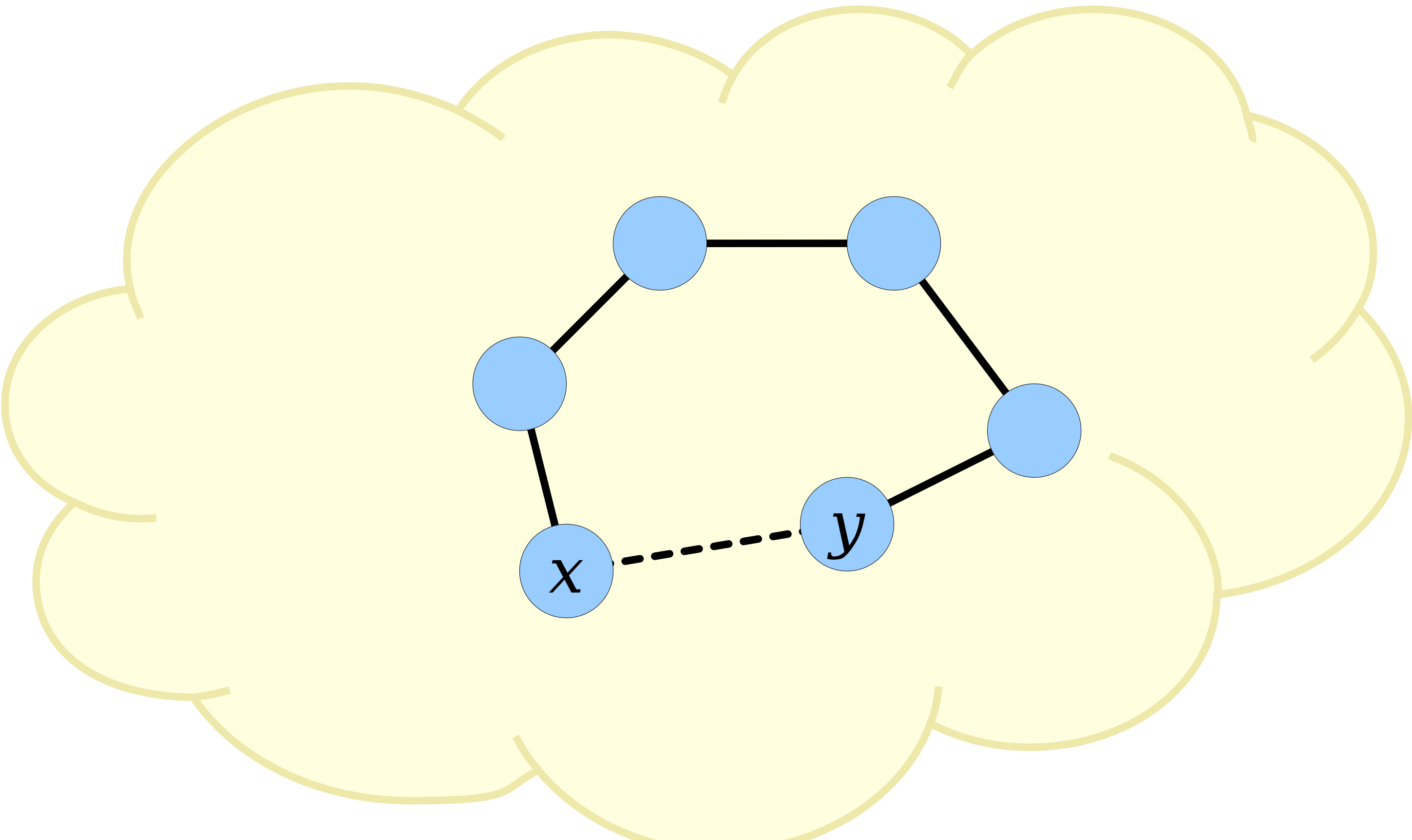

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Check the appendix for the other two steps of the proof.

More to Explore

- A tree kind of seems like a bad way to design a network. (Why?)
- Actual local area networks allow for cycles. They use something called the ***spanning tree protocol*** (***STP***) to selectively disable links to form a tree.
- Routing through the full internet – not just within a LAN – is a fascinating topic in its own right.
- Take CS144 (networking) for details!
- If we have time, we'll explore more on network routing later in the quarter.

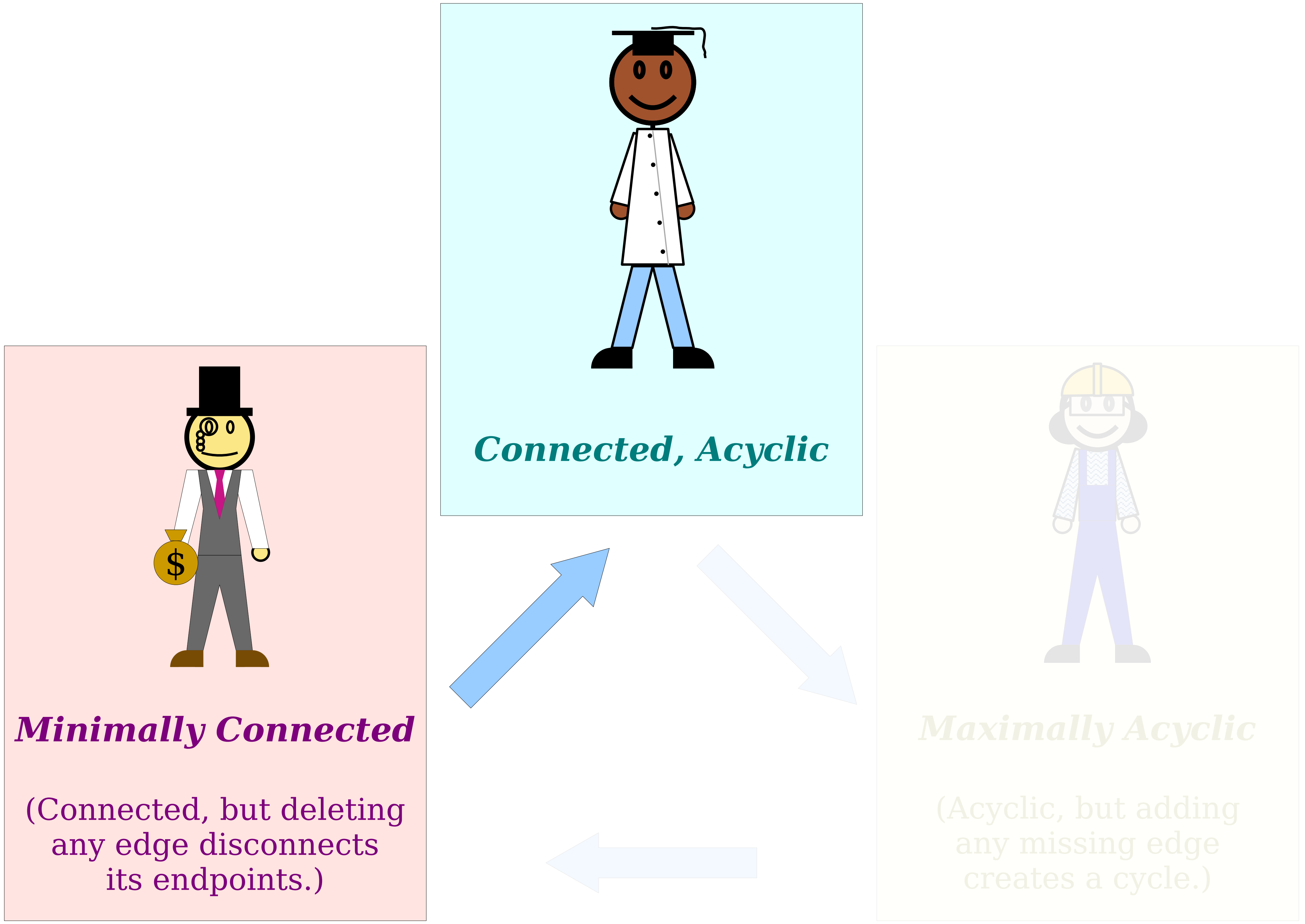
Recap from Today

- ***Walks*** and ***closed walks*** represent ways of moving around a graph. ***Paths*** and ***cycles*** are “redundancy-free” walks and cycles.
- ***Trees*** are graphs that are connected and acyclic. They’re also minimally-connected graphs and maximally-acyclic graphs.
- Trees have applications throughout CS, including networking.

Next Time

- ***The Pigeonhole Principle***
  - A simple, powerful, versatile theorem.
- ***Graph Theory Party Tricks***
  - Applying math to graphs of people!
- ***A Little Movie Puzzle***
  - Who watched what?

Appendix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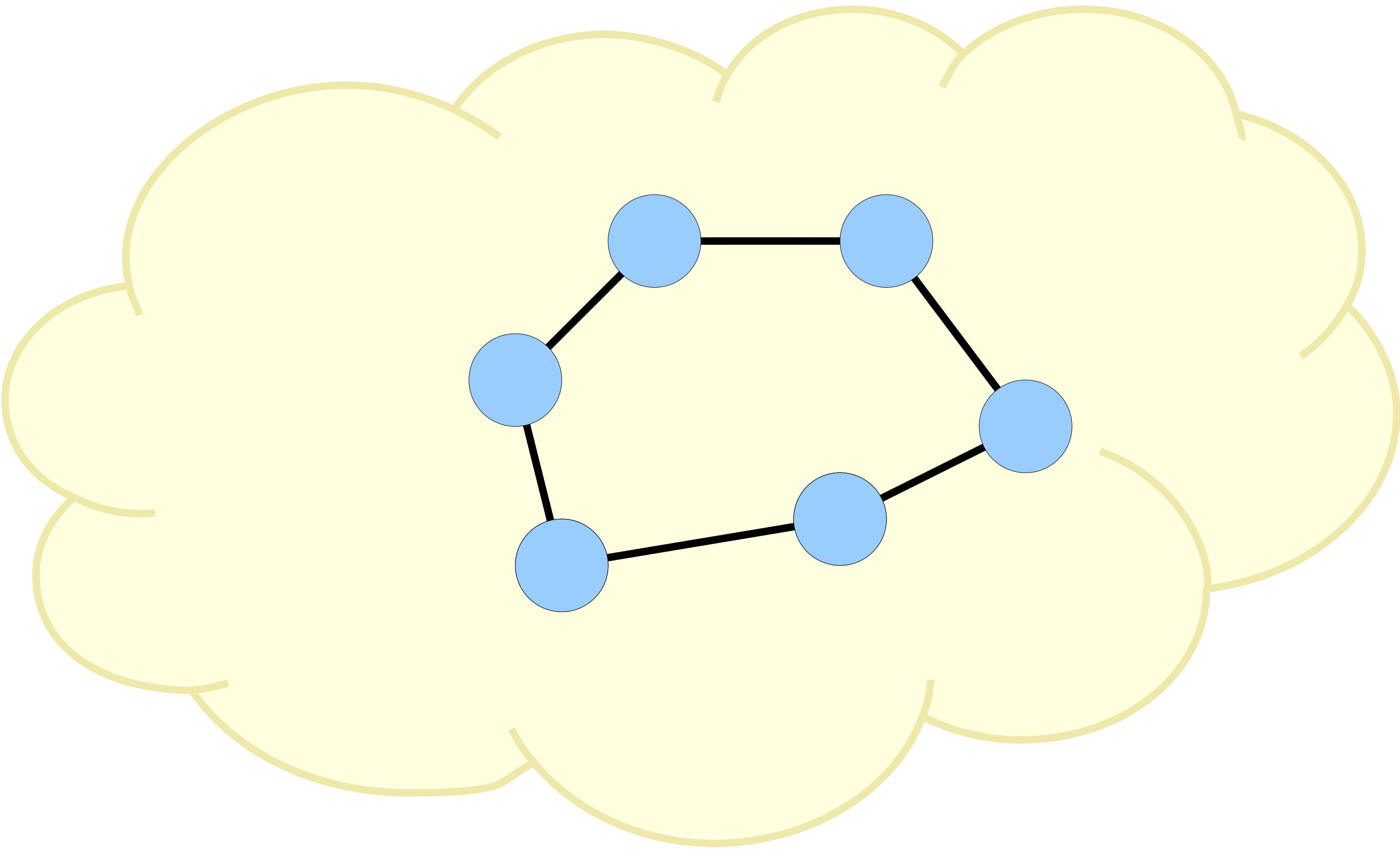

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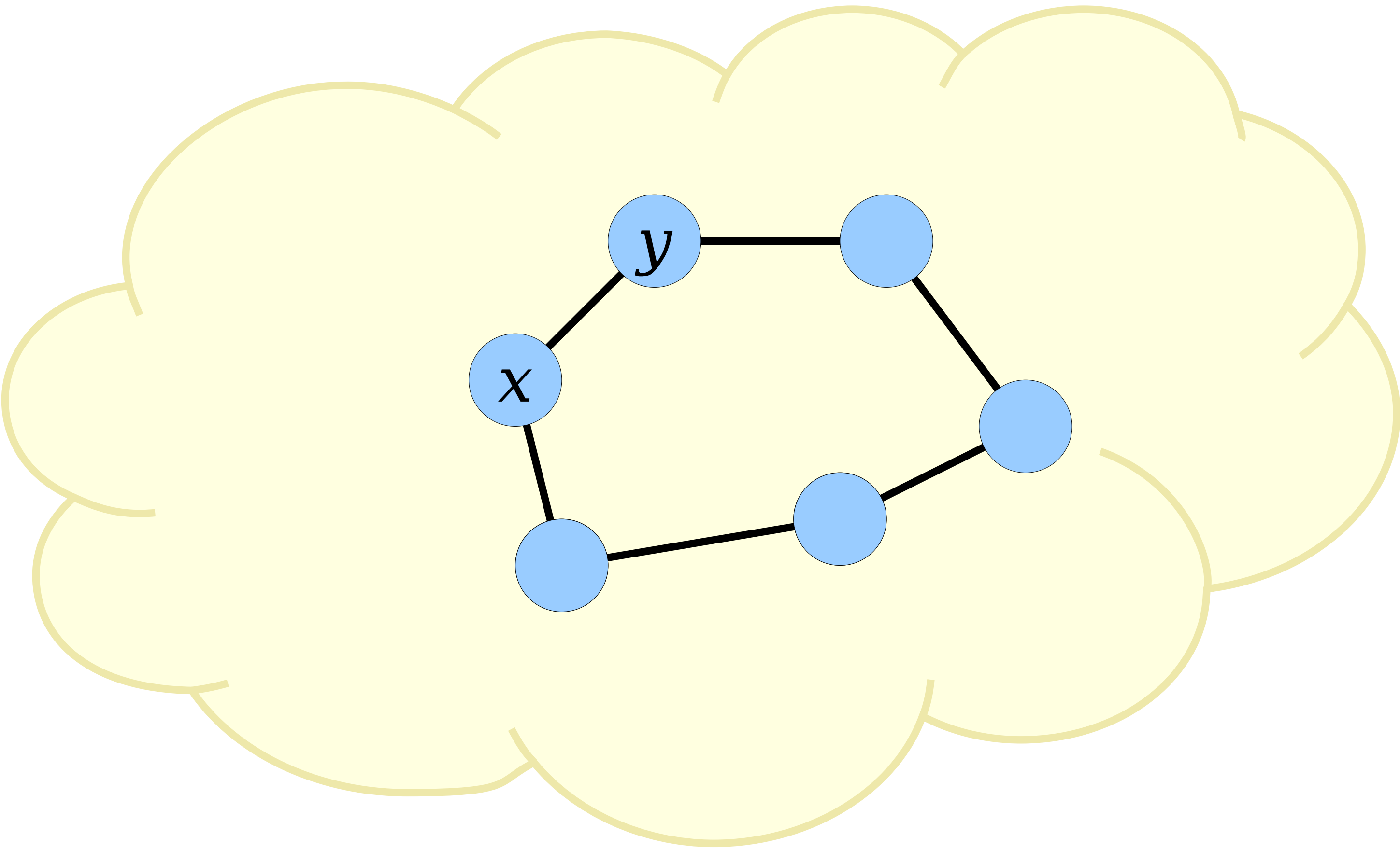

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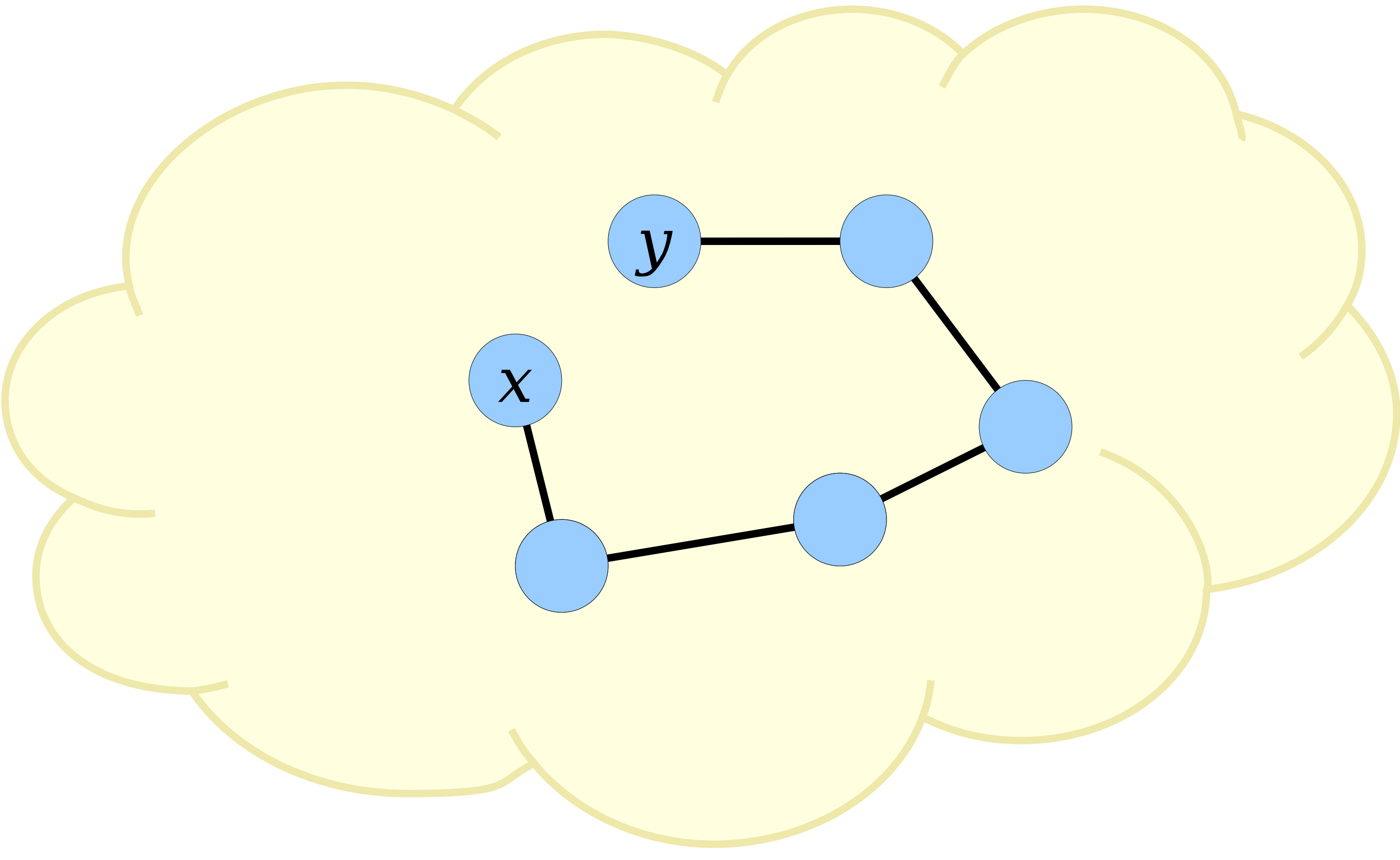

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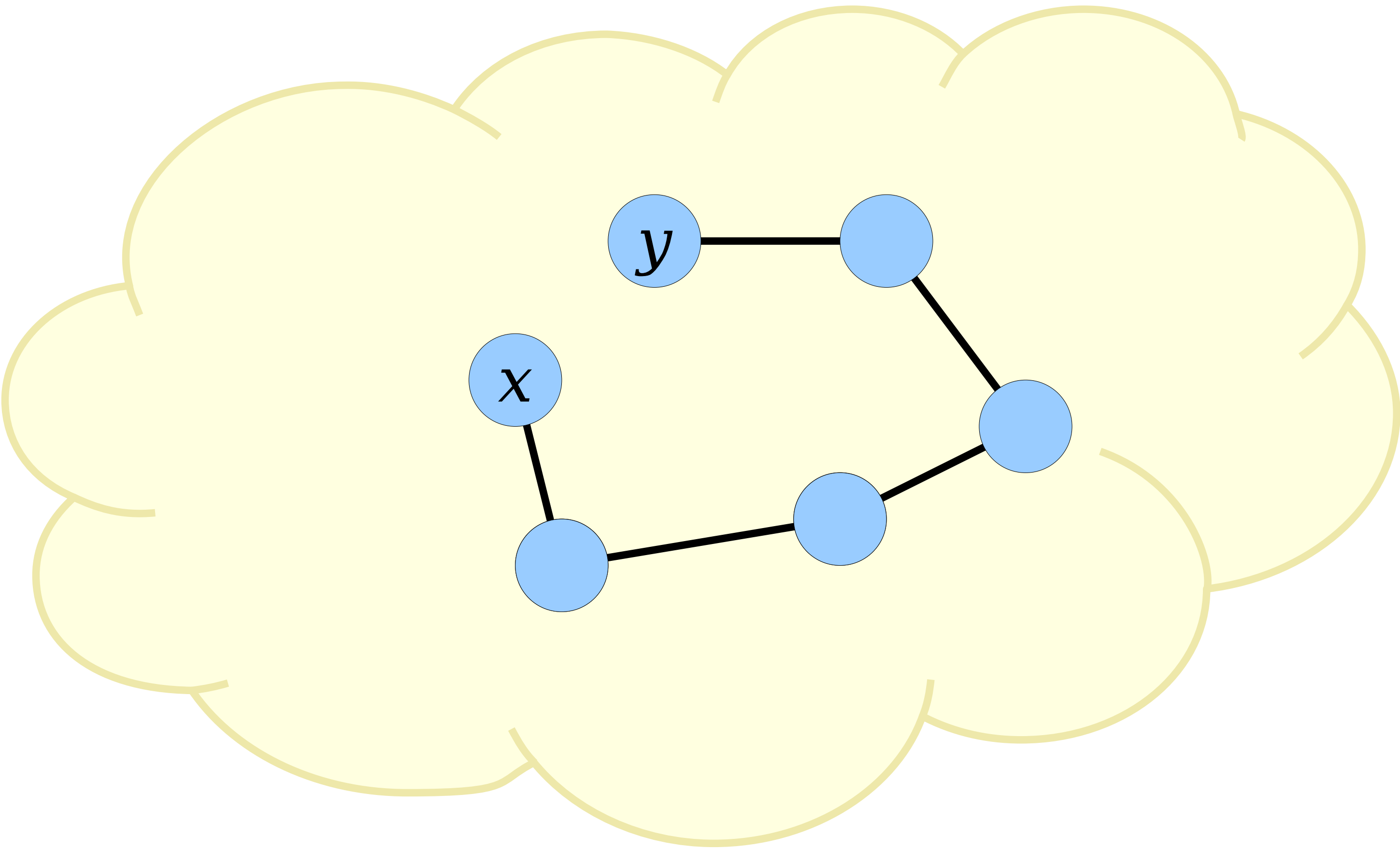

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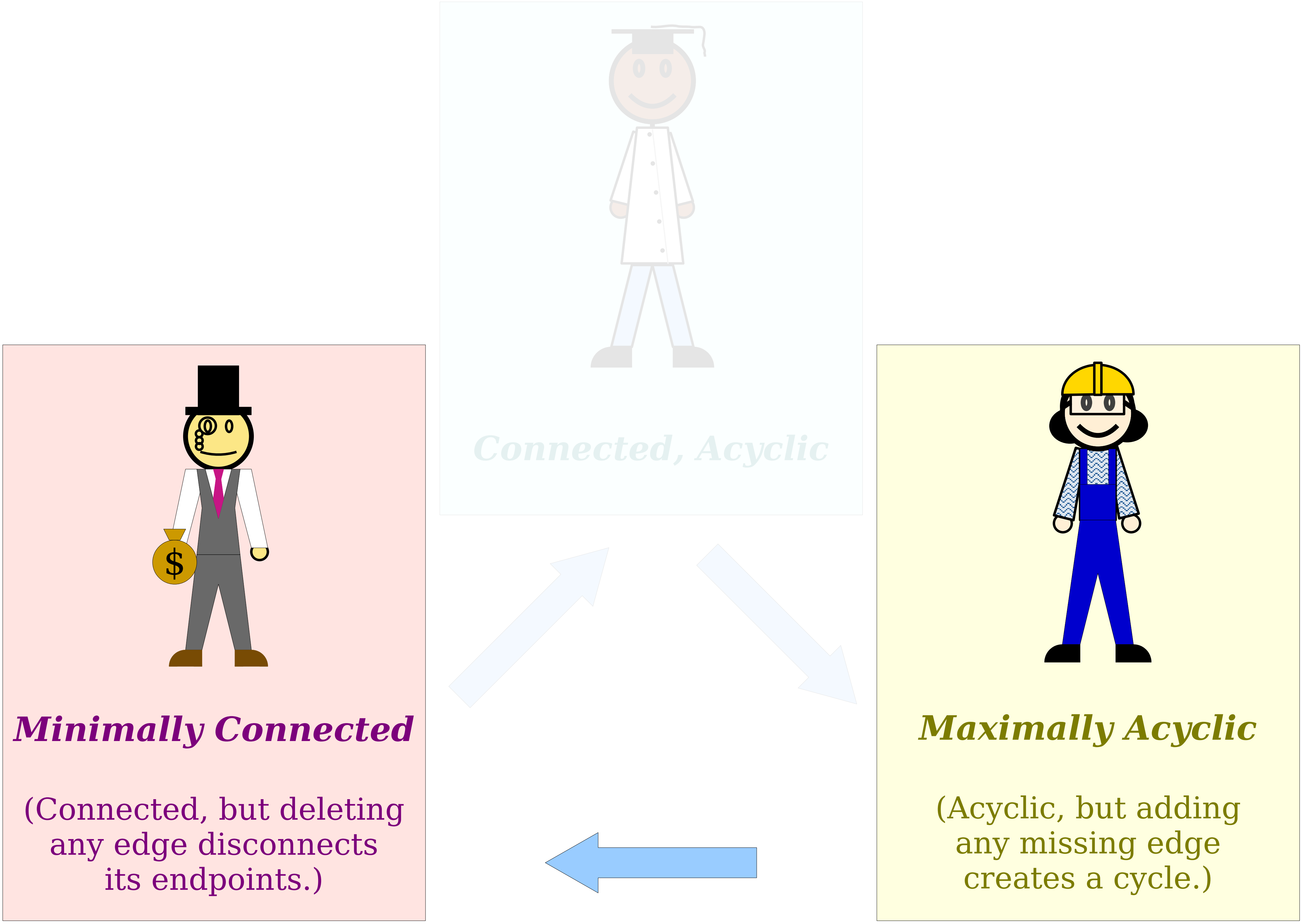
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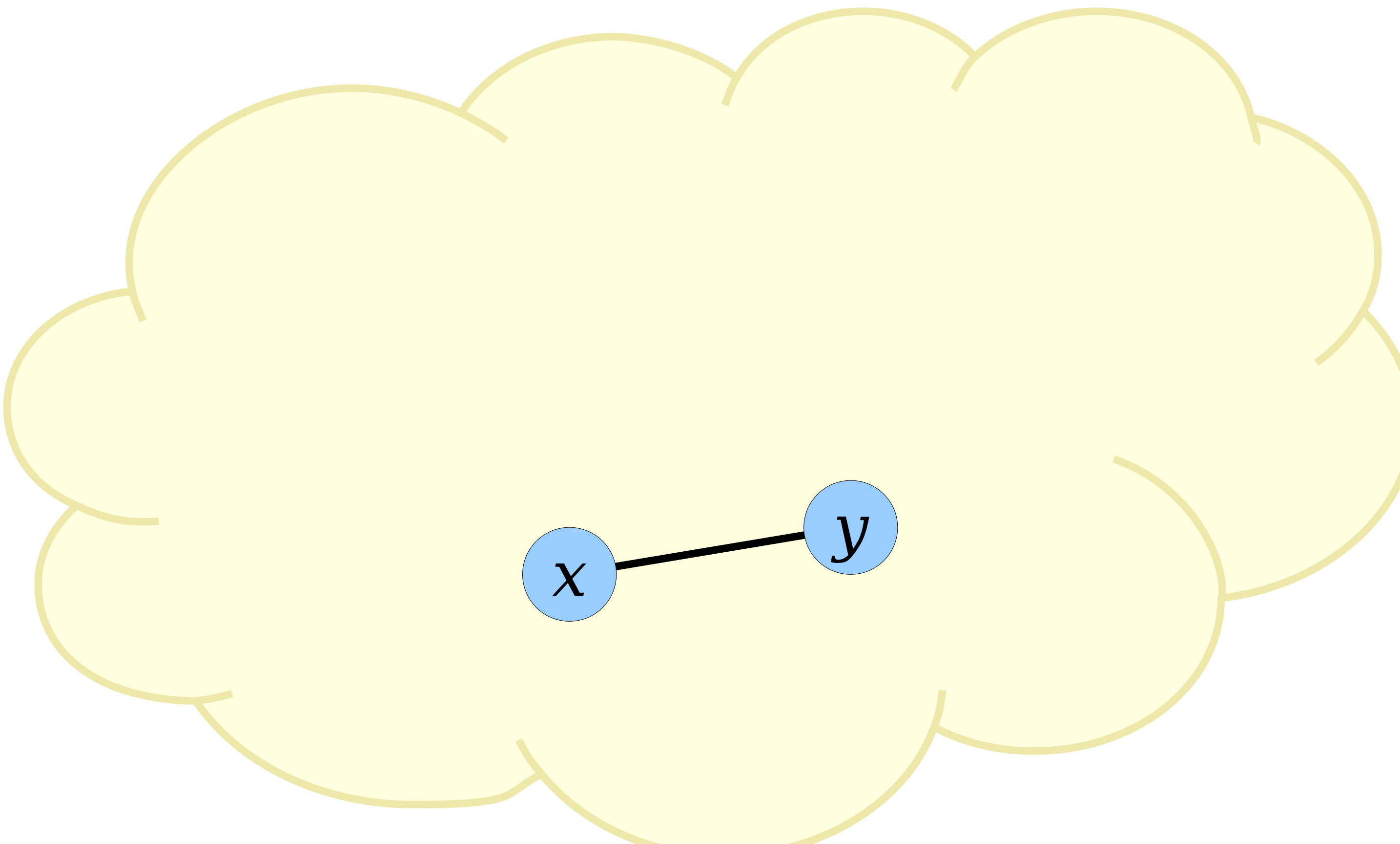

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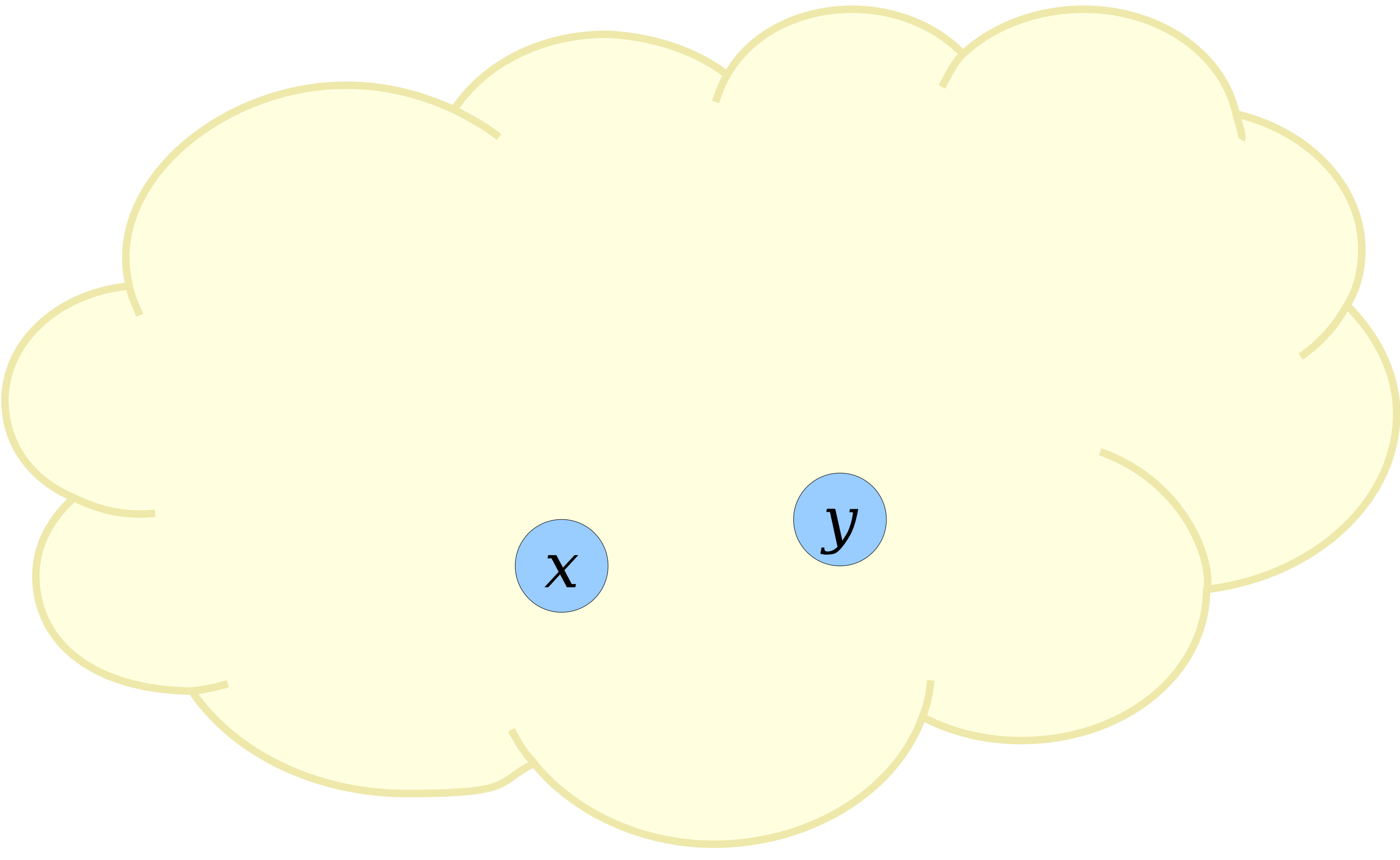

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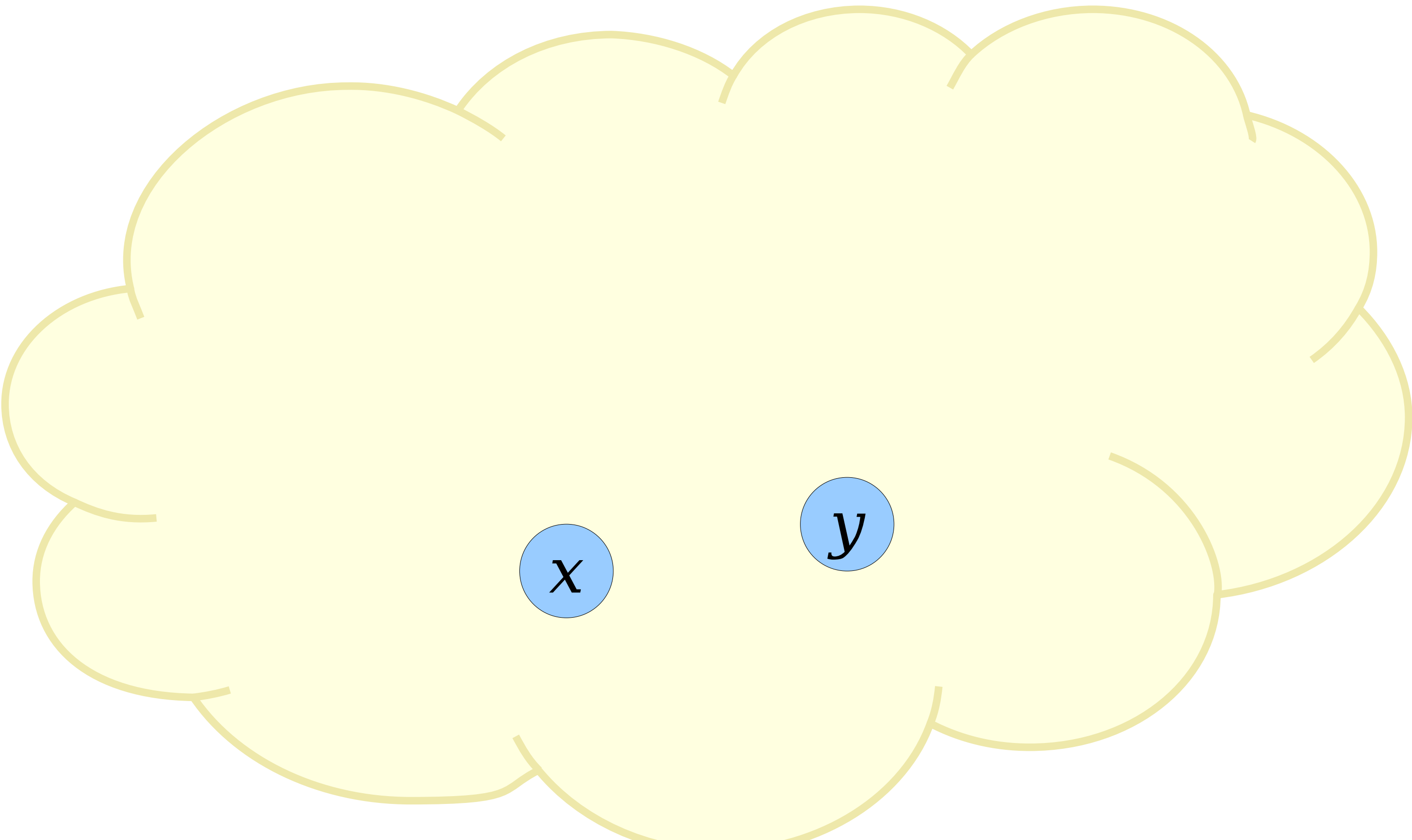

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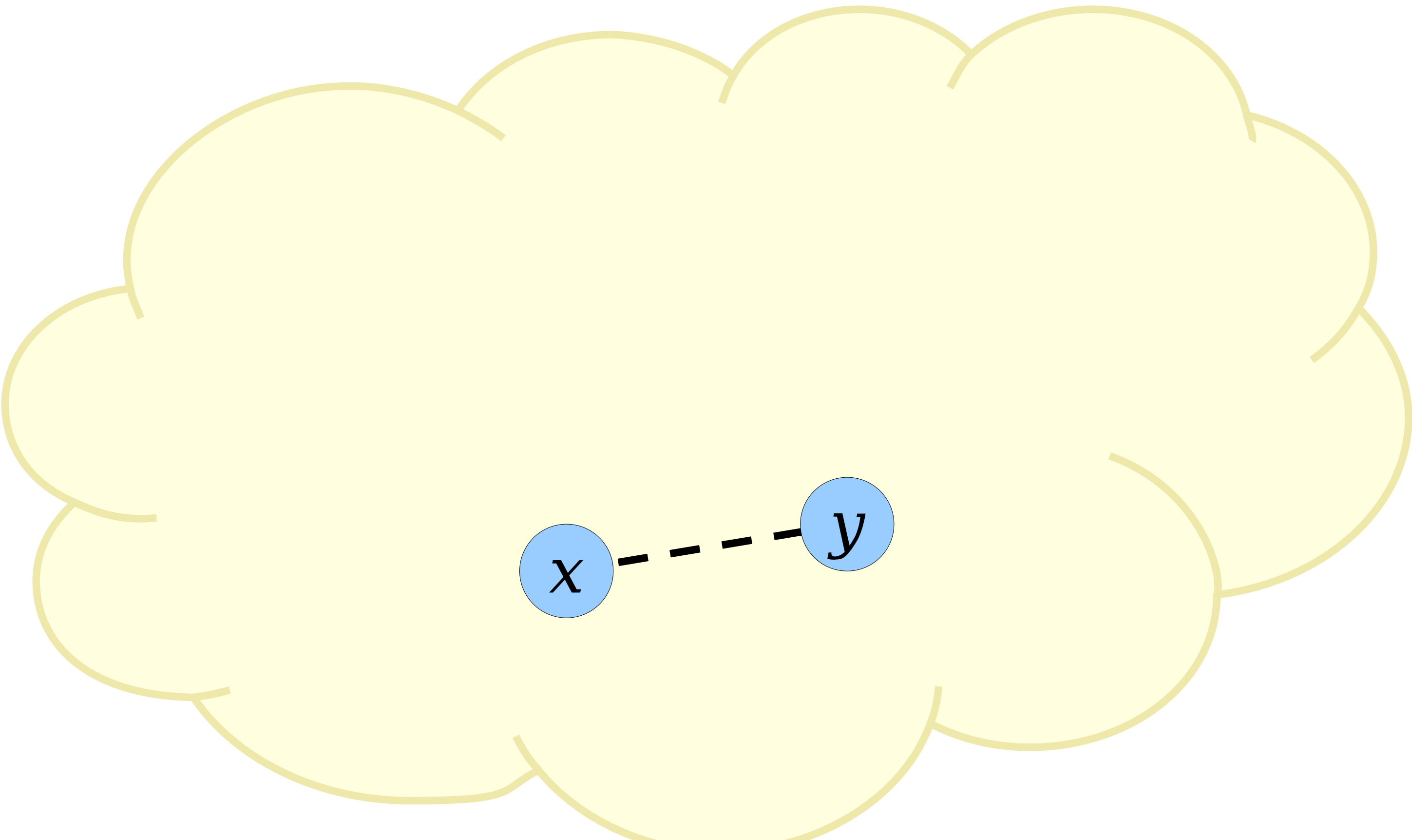

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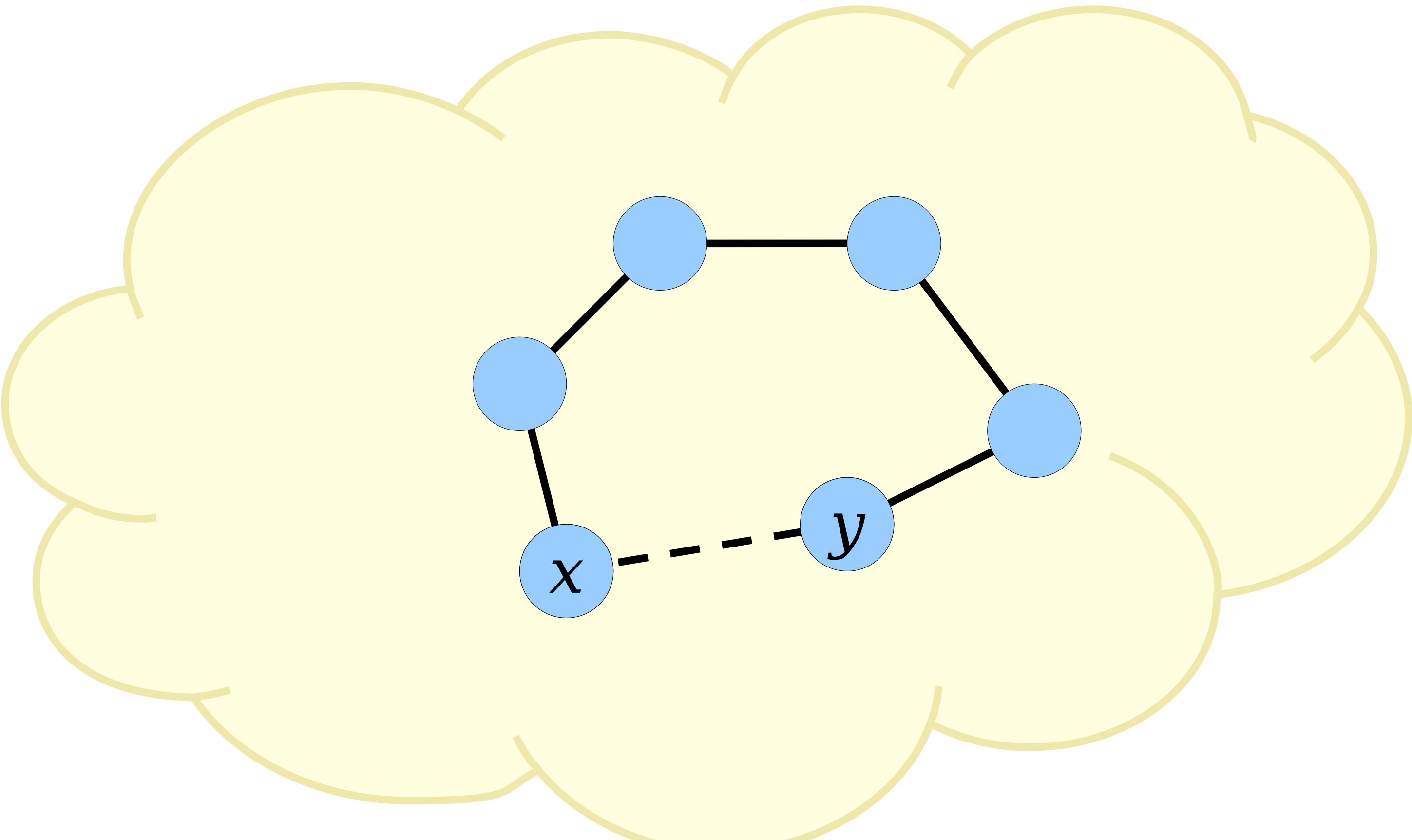

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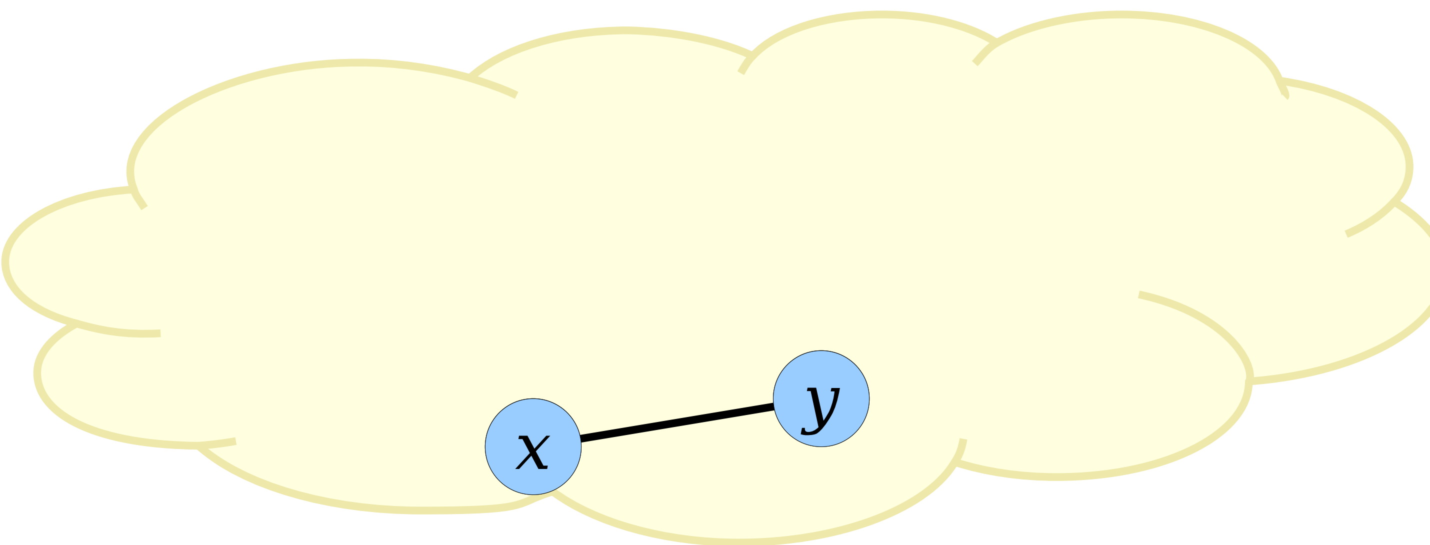

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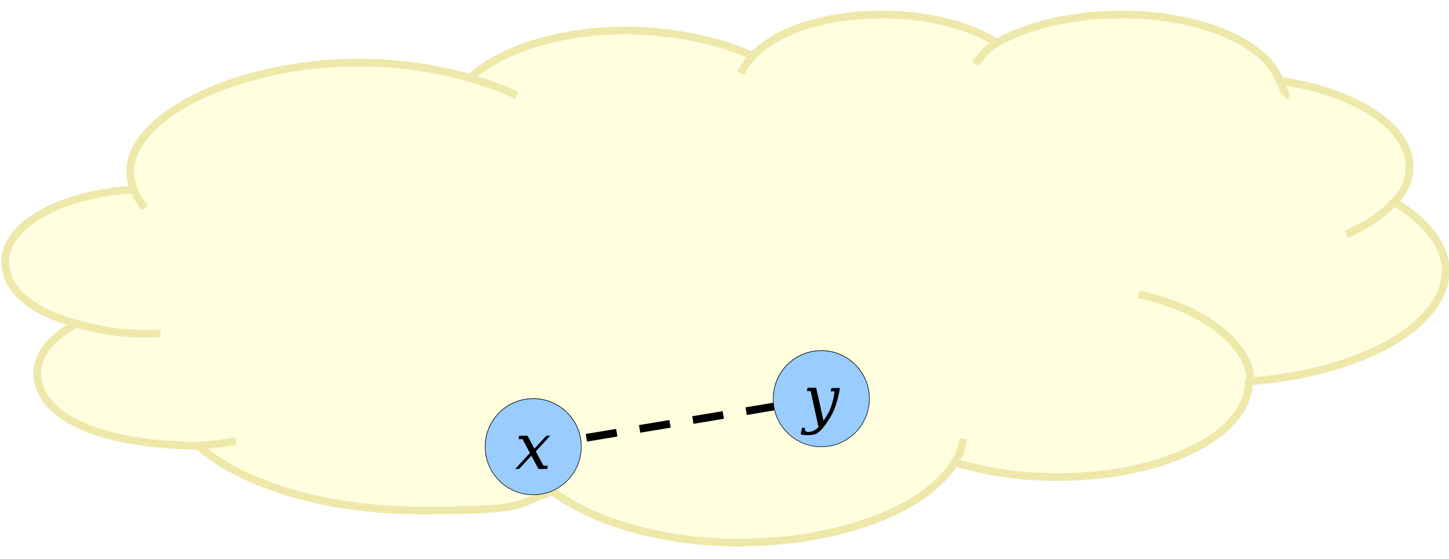

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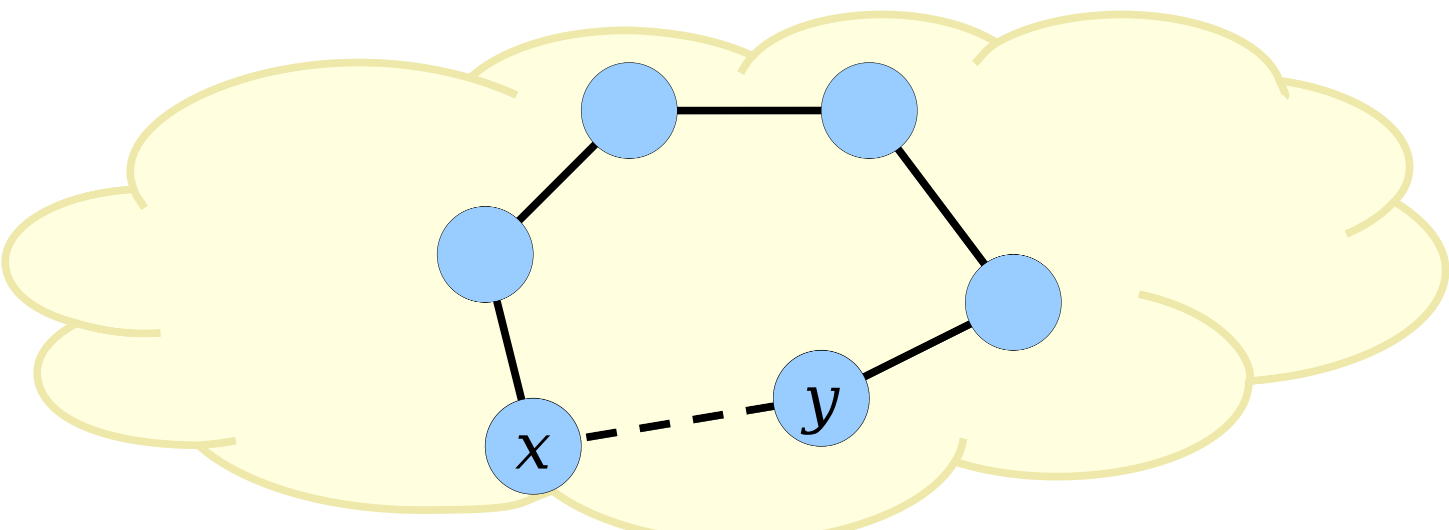

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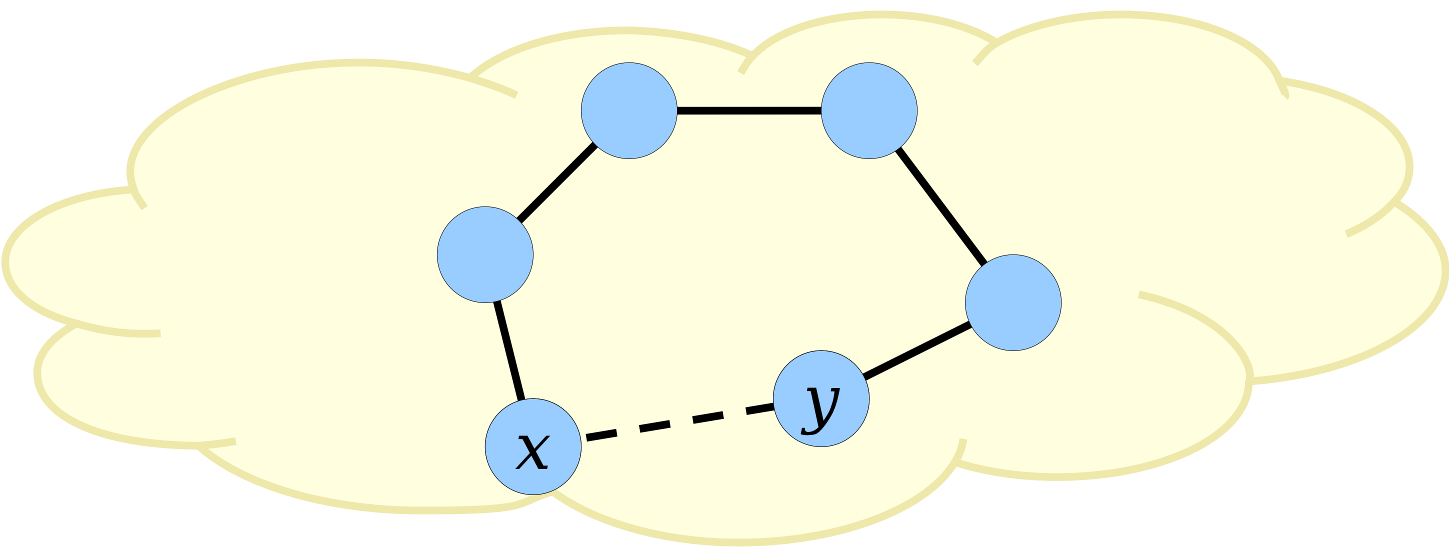

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